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THE EARLY LIFE OF RUSSELL'S NOTION OF A PROPOSITIONAL FUNCTION

ABSTRACT: In this paper I describe the birth of Russell's notion of a propositional function on 3 May 1902 (precise time of day unknown) and its immediate context and implications. In particular, I consider its significance in relation to the development of his views on analysis.

1. INTRODUCTION

One of the key texts in the development of analytic philosophy is the book Russell published in 1903, *The Principles of Mathematics*. The notion of a propositional function makes its first appearance in this book and is central to Russell's philosophy thereafter. In chapter 2 of Part I, in which Russell explains the new symbolic logic that he had learnt from Peano and extended, it is introduced as one of the three fundamental notions that Russell takes to characterize the class-calculus (cf. 1903, p. 19). A more detailed account is provided in chapter 7 of Part I, which bears the title 'Propositional Functions'. The index to the book shows that there is talk of propositional functions elsewhere in the text, and a reader may easily gain the impression that the notion is not only fundamental to the book but also well integrated into the argument.

In fact, however, the notion of a propositional function is only introduced at a very late stage in the composition of the *Principles*, just

weeks before the whole manuscript was submitted to Cambridge University Press in late May 1902. The stage may have been set for its introduction, but given the importance of the notion, the precise nature of its appearance deserves elucidation. I offer an account of its emergence in section 2. My interest in this is not merely antiquarian, however. The evolution of Russell's conception of a propositional function plays a central role in the development of his thinking about analysis, and explaining this thinking is important not only in understanding Russell's philosophy in all its changes but also in appreciating the complex nature of the practices of analysis in analytic philosophy today. I indicate the significance of Russell's notion of a propositional function in the development of his views on analysis in section 3. First, though, in setting the scene, I say a few words about the different forms of analysis to be found in Russell's work.

2. FORMS OF ANALYSIS IN RUSSELL'S WORK

From the time of his rebellion against British idealism, Russell saw analysis as central to his methodology. In the opening chapter of *My Philosophical Development*, for example, he wrote:

> Ever since I abandoned the philosophy of Kant and Hegel, I have sought solutions of philosophical problems by means of analysis, and I remain firmly persuaded, in spite of some modern tendencies to the contrary, that only by analysing is progress possible. (1959, p. 11)

It is far from clear from this, though, what Russell meant by 'analysis', and different conceptions are reflected in both his practice and his methodological remarks. In the secondary literature, it is all too readily assumed that Russell had one particular method in mind, or that there is one particular method that is exemplified in his practice. This assumption is often made in arguing that there is unity and continuity in Russell's philosophy.[1] In my view, however, we should see the changes in Russell's philosophy as reflecting corresponding changes in his conception and practice of analysis. This is not to deny that there is continuity, but I think it is important to appreciate the complexity of Russell's methodology.

If a general characterization of analysis is possible, then it might best be described as "a process of isolating or working back to what is more fundamental by means of which something, initially taken as given, can be explained or reconstructed".[2] But this is too general to be of much use in exploring the variety and richness of the methods and practices of nalysis to be found in the history of philosophy. A better conceptual framework is provided by distinguishing three primary *modes* of analysis, as I call them – the *regressive*, the *decompositional*, and the *interpretive* (or *transformative*). Briefly stated, the regressive mode involves working back to the principles, premises, causes, etc., by means of which something can be derived or explained, the decompositional mode involves identifying the elements and structure of something, and the interpretive mode involves 'translating' something into a particular framework. These modes may be realized and combined in a variety of ways, in constituting specific conceptions or practices of analysis. Where one mode is dominant in a given conception, we may talk, for example, of the decompositional conception; but it should be stressed that in actual practices of analysis, all three modes are typically combined.[3]

The regressive mode originated in ancient Greek geometry, and has formed the core of a common conception ever since. A classic statement was given by Pappus in his *Mathematical Collection* around 300 AD: "In analysis we suppose that which is sought to be already done, and we inquire from what it results, and again what is the antecedent of the latter, until we on our backward way light upon something already known and being first in order."[4] This conception was central in discussions of Aristotelian methodology during the Renaissance, and also found expression in the Port-Royal *Logic* of the seventeenth century. We can see it illustrated, too, in Frege's and Russell's logicist project – in seeking to identify the fundamental logical axioms and definitions by means of which to prove mathematical truths. The conception was articulated by Russell himself in a paper he wrote in 1907 entitled 'The Regressive Method of Discovering the Premises of Mathematics'.[5]

Although the decompositional mode is also exhibited in ancient Greek geometrical analysis, it rose to prominence during the early modern period, inspired by Descartes' work in analytic geometry. It achieved its philosophically most significant form in the decompositional conception of conceptual analysis developed by Leibniz and Kant. Central to Leibniz's philosophy was what can be called his *containment principle*: "in every affirmative true proposition, necessary or contingent, universal or singular, the notion of the predicate is contained in some way in that of the subject, *praedicatum inest subjecto*" (1973, p. 62). Analysis was then seen as the process of decomposing the subject concept into its constituent concepts until the containment of the relevant predicate is explicit, thereby achieving a proof of the proposition. Although Kant came to reject the generality of Leibniz's view, he accepted that containment held the key to what he called 'analytic' truths. A true proposition of the form 'a is B' is 'analytic', on Kant's account, if and only if the predicate B is contained in the subject a.[6]

The decompositional conception of analysis has dominated philosophy in the modern period, from Descartes onwards. Although Kantian (along with Hegelian) philosophy was rejected by Russell and Moore in their early work, they retained the underlying conception of analysis. Indeed, their rebellion against British idealism was grounded on their endorsement of decompositional analysis as the primary method of philosophy. This endorsement can be seen as one characteristic feature of the 'analytic' tradition that they helped found. But precisely because decompositional analysis was not itself new, this is hardly sufficient to explain what was ground-breaking about analytic philosophy. On my view, it is the role played, instead, by interpretive or transformative analysis that is particularly distinctive of analytic philosophy, or at least, of one central strand in analytic philosophy, and it is the interpretive mode of analysis that came of age in early twentieth-century philosophy.

Interpretive analysis was not new in analytic philosophy. On the contrary, it is implicit in all forms of analysis. For in attempting to analyze anything, we need first to interpret it in some way – drawing a diagram, for example, or 'translating' an initial statement into the privileged language of logic, mathematics or science – in order for the resources of a relevant theory or conceptual framework to be brought to bear. In Euclidean geometry, for example, a diagram is typically provided in order to see exactly what is to be demonstrated and to provide the basis for adding the auxiliary lines that are needed in the required construction or proof. In analytic geometry, geometrical problems are first 'translated' into the language of arithmetic and algebra in order to solve them more easily. Indeed, in the sixteenth and seventeenth centuries, algebra was specifically called an 'art of analysis', and while this phrase was used in deliberate allusion to the supposed secret art of analysis of the ancients, it is also appropriate, as I see it, in light of the role played by interpretive analysis.[7] Descartes himself may have conceived of analysis in decompositional terms, as illustrated, for example, in his stressing how complex problems should be broken down into simpler ones.[8] But that does not mean that interpretive analysis played no role in his practice. Indeed, the superior power of analytic geometry as opposed to synthetic geometry – as Euclidean geometry then came to be called by contrast – lies precisely in the translation of geometrical problems into arithmetical and algebraic ones, allowing the richer resources of arithmetic and algebra to be utilized.

When we come to analytic philosophy, the significance of interpretive analysis is revealed most clearly in logical formalization. Just as in analytic geometry the problems are translated into the language of arithmetic and algebra to facilitate their solution, so too in analytic philosophy – or at least, in that central strand originating in the work of Frege and Russell – the statements seen as philosophically problematic are translated into the language of logic to reveal their 'real' logical form. If this analogy is right, then analytic philosophy is 'analytic' much more in the sense that analytic geometry is analytic than in the crude decompositional sense that many have taken 'analysis' to have.

Russell's theory of descriptions has frequently been described as a paradigm of philosophical analysis. On my view, though, it is a paradigm of interpretive analysis rather than of decompositional analysis – or more exactly, of the role that interpretive analysis plays as a prerequisite to decompositional analysis. For the key move here is translating or paraphrasing a problematic statement (in this case, a statement in which there is apparent reference to something that does not exist) into one which makes clearer what is meant, or what ontological commitments are involved. As Russell recognized, it was the invention of quantificational theory, which provided a far more powerful interpretive system than anything that had hitherto been available, that made such logical translation possible and that truly inaugurated the age of analytic philosophy.

The theory of descriptions, however, did not appear until 1905, and the story of Russell's earlier work is indeed one in which decompositional analysis is dominant. What happens, though, is that Russell gradually realizes the limitations of decompositional analysis and develops ever more sophisticated analytic tools – such as logical formalization and later logical construction – to supplement his use of decompositional analysis. It is this move that I want to illustrate in the present paper by considering the early life of his notion of a propositional function. Russell never abandons his belief that decompositional analysis lies at the core of his method, but his actual practice certainly shows that more subtle conceptions gradually enter the picture.

On Russell's early view, then, analysis is understood as the process of breaking something down into its constituents – or at the intellectual level, as the process of identifying the constituents of something. This is reflected in the centrality that the whole-part relationship had in his thinking during this period. As Russell wrote in his 1899/1900 draft of the *Principles*, "The only kind of unity to which I can attach any precise sense—apart from the unity of the absolutely simple—is that of a whole composed of parts".[9]

In his initial rebellion against British idealism, Russell adopted a view of propositions as being both independent of the mind and quite literally composed of their constituents. In adopting this view, Russell was influenced by Moore, who, in his own early work, had declared that "a proposition is nothing other than a complex concept ... a synthesis of concepts" (1899, p. 5). According to Moore, concepts are the things out of which the world itself is formed, leading naturally to the claim that "A thing becomes intelligible first when it is analysed into its constituent concepts" (1899, p. 8). A decompositional conception of analysis is clearly presupposed here.

The significance of this conception can be illustrated by taking the case of relational propositions, which, as Russell recognized from the very beginning, are fundamental in mathematics. Relational propositions are constituted, on Russell's view, by the relations and relata they contain. Consider the following example that Frege gives in §9 of his *Begriffsschrift* of 1879:

(Lhc) Hydrogen is lighter than carbon dioxide.

According to Frege, this has the same 'content' as:

(Hch) Carbon dioxide is heavier than hydrogen.

We might agree with Frege that there is at least some sense in which (Lhc) and (Hch) have the same 'content'; certainly, they are equivalent in the sense that if one is true, then the other is true, and vice versa. At the time of the *Principles*, however, Russell regarded these as representing two *different* propositions, because while one proposition contains the relation *is lighter than*, the other contains the converse relation *is heavier than*, and these are different relations. What is driving his departure from Frege (and perhaps our own intuitions about sameness of content) here is the idea that a proposition is quite literally composed of its constituents.[10]

A proposition, however, cannot be treated as simply a collection of constituents. Consider the following:

(Lch) Carbon dioxide is lighter than hydrogen.

On Russell's view, the propositions expressed by (Lhc) and (Lch) have exactly the same constituents – the objects hydrogen and carbon dioxide and the relation *is lighter than*. Yet they are clearly different propositions: one is true and the other is false. Russell took this to show that something was therefore right in the Bradleian doctrine that (crude decompositional) analysis is falsification. In the case of propositions, the whole is more than the sum of its parts, and breaking it down into its constituents loses something essential. Nevertheless, he insisted on the importance of analysis. In discussing the doctrine in the *Principles*, he wrote:

> In short, though analysis gives us the truth, and nothing but the truth, yet it can never give us the whole truth. This is the only sense in which the doctrine is to be accepted. In any wider sense, it becomes merely a cloak for laziness, by giving an excuse to those who dislike the labour of analysis.[11]

The obvious answer to the objection that analysis is falsification, however, is to argue that analysis must identify not only the constituents of a proposition but also the *way* in which they are arranged, i.e., the *form* of the proposition. Russell recognized that a distinction needed to be drawn between two types of wholes, which he called 'aggregates',

which are simply the sum of their parts, and 'unities', which are more than the sum of their parts (1903, ch. 16). But he did not immediately take this to suggest two notions of analysis, or a richer notion that construes analysis as the identification of the form as well as the constituents of a proposition. This is something that developed later, and received its most explicit statement in the chapter on analysis and synthesis in his 1913 'Theory of Judgement' manuscript.[12] Russell distinguishes here between 'material' and 'formal' analysis, both being reflected in his definition of 'analysis' (in the broadest decompositional sense) as "the discovery of the constituents and the manner of combination of a given complex".[13]

Understanding how form is involved here is an important element in Russell's struggle with the problem of the unity of the proposition.[14] As he recognized from the very beginning, a proposition is not just a collection of constituents, but has an essential 'unity'. Such a unity can be seen as conferred by the *way* in which those constituents are arranged, reflecting the *form* of the proposition. Introducing talk of 'forms' opens up a whole new dimension to analysis: not just identifying the relevant forms, as if that were a relatively straightforward matter, but elucidating the many complex ways in which different kinds of things fit together. As Russell thought through the implications of his new philosophical views and concerns, in other words, there was an inevitable move away from the crude decompositional conception of analysis that he initially endorsed in his rebellion against British idealism.

The limitations of this conception are obvious when we consider function-argument analysis, on which quantificational logic depends: in general, the value of a function does not literally contain its argument(s) as part(s). Russell began to appreciate the power of function-argument analysis after his meeting with Peano in 1900, and as he learnt, developed and applied Peano's logic, he was forced to rethink his adherence to decompositional (whole-part) analysis. This is illustrated most strikingly in the evolution of his conception of a propositional function, which came to lie at the core of his philosophy. The idea of a propositional function is introduced very early in the final text of the *Principles*, but it did not emerge until relatively late in the actual writing of the text, and Russell's conception continued to evolve after the completion of the *Principles*. I describe its emergence in the next section, before indicating its implications and subsequent development in section 3.

3. THE BIRTH OF THE NOTION OF A PROPOSITIONAL FUNCTION

We can begin here with the draft that Russell wrote some time between August 1899 and June 1900, which is the first to appear under the title 'The Principles of Mathematics'.[15] Part I is called 'Number', beginning with a chapter on 'Collections', and Part II is called 'Whole and Part', beginning with a chapter on 'The Meaning of Whole and Part', where we find the first draft of the remark quoted above about analysis not giving us "the whole truth".[16] There is no mention of propositional functions, which is not surprising, given that Russell had not yet met Peano. In the months immediately following his meeting with Peano (from October to December 1900), however, he redrafted Parts III to VI, which became the corresponding Parts of the *Principles*, with only minor subsequent revision. He did not return to Parts I and II until May and June 1901, when Part I on 'Number' became Part II (as it is in the *Principles*), what had been Part II was reworked into this new Part II, and a completely fresh Part I was written.

This fresh version of Part I is significant because it represents Russell's first attempt to explain the logical system he had learnt from Peano and developed for his own purposes. Entitled 'The Variable', it gives pride of place to what Russell saw as the central notion of logic, the 'logical variable', understood as ranging over everything, and not just the objects of mathematics (cf. 1901, p. 201). Even here, however, there is still no talk of propositional functions. The closest Russell comes is in the draft chapter on 'Peano's Symbolic Logic'. Russell writes:

> . . . what Peano and mathematicians generally call *one* proposition containing a variable is really the disjunction or conjunction (according to circumstances) of a certain class of propositions defined by some constancy of form. . . . "x is an a" stands for the variable conjunction of all true propositions in which terms are said to be a's; and similar remarks apply to any other proposition containing a variable. (1901, p. 205)

'x is an a', where x is the variable, is what comes to be called a 'propositional function'.

Russell goes on to note that 'x is an a' can be symbolized as '$f(x)$', but he recognizes that there is a difficulty in regarding what is represented here as a (single) proposition:

> . . . it must always be remembered that the appearance of having *one* proposition $f(x)$ satisfied by a number of values of x is fallacious: the proposition $f(x)$ is a conjunction of just as many propositions as there are terms in the class of terms such that $f(x)$ is true. (1901, p. 206)

Russell senses the problem lurking here, but he does not deny that what we have is a proposition: it is seen, though, not as a single proposition but as a conjunction of propositions, i.e., a complex proposition. Compare this with the corresponding passage in the *Principles*:[17]

> . . . it must always be remembered that the appearance of having *one* proposition $f(x)$ satisfied by a number of values of x is fallacious: $f(x)$ is not a proposition at all, but a propositional function. (1903, p. 29)

The confusions that may arise in regarding 'x is an a' as representing a proposition are now removed: what we have is a propositional function.

We can date the emergence of the notion of a propositional function in Russell's work to the first week of May 1902. The notion is not referred to in the plan for a revised version of Part I that he drew up at the very end of April 1902,[18] yet the revision of that Part – in which the notion finally appears – was completed by 13 May.[19] So the idea clearly came to Russell at some point during this period. In fact, we can be far more precise than this. In the *Principles*, the notion is introduced at the beginning of chapter 2 (p. 13), and identified as a fundamental notion a few pages later (p. 19). The passage quoted above also comes from this chapter, towards the end. Chapter 2 was written (almost certainly) on 3 and 4 May.[20] So this makes 3 May the most likely date of birth of Russell's notion of a propositional function.

If we look at the passage in which the notion of a propositional function is introduced, it is clear what motivated it. The passage occurs in section 13 of the book, the third section of chapter 2, the chapter entitled 'Symbolic Logic', where Russell explains the new logical system he had learnt from Peano and had been developing and applying. Russell is concerned in this section with the distinction between the calculus of propositions and the calculus of classes, and sides with McColl in seeing the former as more fundamental.[21] But he criticizes McColl for not appreciating the distinction between "genuine propositions and such as contain a real variable", which leads McColl, according to Russell, "to speak of propositions as sometimes true and sometimes false, which of course is impossible with a genuine proposition" (1903, p. 12). Russell goes on:

> As the distinction involved is of very great importance, I shall dwell on it before proceeding further. A proposition, we may say, is anything that is true or that is false. An expression such as "x is a man" is therefore not a proposition, for it is neither true nor false. If we give to x any constant value whatever, the expression becomes a proposition: it is thus as it were a schematic form standing for any one of a whole class of propositions. And when we say "x is a man implies x is a mortal for all values of x," we are not asserting a single implication, but a class of implications; we have now a genuine proposition, in which, though the letter x appears, there is no real variable: the variable is absorbed in the same kind of way as the x under the integral sign in a definite integral, so that the result is no longer a function of x. Peano distinguishes a variable which appears in this way as *apparent*, since the proposition does not depend upon the variable; whereas in "x is a man" there are different propositions for different values of the variable, and the variable is what Peano calls *real*. I shall speak of propositions exclusively where there is no real variable: where there are one or more real variables, and for all values of the variables the expression involved is a proposition, I shall call the expression a *propositional function*. The study of genuine propositions is, in my opinion, more fundamental than that of classes; but the study of propositional functions appears to be strictly on a par with that of classes, and indeed scarcely distinguishable therefrom. Peano, like McColl, at first regarded propositions as more fundamental than classes, but he, even more definitely, considered propositional functions rather than propositions. (1903, pp. 12-13)

The notion of a propositional function thus emerges in articulating the distinction between 'genuine propositions', the bearers of truth or falsity, and schematic expressions that represent a class of propositions, expressions which do so by containing one or more 'real variables', to use the term Russell took from Peano. Since these expressions yield genuine propositions whenever all their real variables are given values, it is natural to call them 'propositional functions'.[22] A few pages later, in claiming that the notion of a propositional function is one of the three fundamental notions that characterize the class-calculus (alongside the notions of class-membership and of *such that*), Russell explains it as follows:

> φx is a propositional function if, for every value of x, φx is a proposition, determinate when x is given. Thus "x is a man" is a propositional function. In any proposition, however complicated, which contains no real variables, we may imagine one of the terms, not a verb or adjective, to be replaced by other terms: instead of "Socrates is a man" we may put "Plato is a man," "the number 2 is a man," and so on. Thus we get successive propositions all agreeing except as to the one variable term. Putting x for the variable term, "x is a man" expresses the type of all such propositions. (1903, pp. 19-20)

After its introductory sections (§§ 11-13), chapter 2 of the *Principles* is divided into four parts, on the propositional calculus (§§ 14-19), the calculus of classes (§§ 20-6), the calculus of relations (§§ 27-30), and Peano's symbolic logic (§§ 31-6). For this final part, Russell simply took over the draft chapter on 'Peano's Symbolic Logic' he had composed a year earlier.[23] Comparison of the two texts reveals little revision.[24] The only significant change is that noted above: with the distinction between genuine propositions and propositional functions now drawn, Russell was able to refine his account of expressions such as 'x is an a'.

Russell's main discussion of propositional functions, however, occurs in chapter 7, which bears the title 'Propositional Functions'. This chapter, it seems, was written on 10 May 1902.[25] In the plan drawn up in April 1902, it was called 'Assertions', and it is clear that the central issue to be addressed was the analysis of propositions. In the published chapter itself, Russell reiterates his concern that "when a proposition is completely analyzed into its simple constituents, these constituents taken together do not reconstitute it" (1903, p. 83). In places, he talks of the analysis of a proposition into subject and 'assertion'. In 'Socrates is a man', he writes, "we can plainly distinguish Socrates and something that is asserted about him; we should admit unhesitatingly that the *same* thing may be said about Plato or Aristotle" (1903, p. 84). An 'assertion', as Russell characterizes it, is "everything that remains of the proposition when the subject is omitted", i.e., what is obtained "by simply omitting one of the terms occurring in the proposition" (1903, pp. 83, 85). An assertion thus counts as a constituent of a proposition, even if there remains a problem in explaining how it contributes to the unity of a proposition.

The connection between an assertion and a propositional function can now be easily seen. For if what is omitted (in obtaining an assertion) is instead replaced by a variable, then we have a propositional function. What seems to have happened during the course of writing chapter 7 is that Russell switched from his intended focus on assertions to further exploration of the notion of a propositional function: given how recently the notion had been introduced, this is hardly surprising, and the new title reflects this change.[26] He begins, indeed, by saying that the task of chapter 7 (in accord, presumably, with his original plan drawn up 10 days or so earlier) is to critically examine the "scope and legitimacy" of the general notion of an assertion and its connection with the notions of class and of *such that* (p. 82). But the focus soon shifts to the question of the definability of propositional functions and this then becomes the central issue of the chapter.[27]

Concern with assertions does not disappear, however. What Russell considers is whether 'propositional function' can be defined in terms of 'assertion', along the lines just indicated. Can a propositional function be seen as an assertion plus one or more variables? This looks plausible in the case of simple subject-predicate propositions such as 'Socrates is a man'. But whether or not this is indeed so, problems certainly arise when we consider more complex propositions, where there is more than one term that can be omitted or replaced by a variable. Russell

takes the proposition 'Socrates is a man implies Socrates is a mortal'. Omitting Socrates here, Russell writes, yields "... is a man implies ... is a mortal". But in this formula, he goes on:

> it is essential that, in restoring the proposition, the *same* term should be substituted in the two places where dots indicate the necessity of a term. It does not matter what term we choose, but it must be identical in both places. Of this requisite, however, no trace whatever appears in the would-be assertion, and no trace can appear, since all mention of the term to be inserted is necessarily omitted. When an x is inserted to stand for the variable, the identity of the term to be inserted is indicated by the repetition of the letter x; but in the assertional form no such method is available. (1903, p. 85)

On Russell's account, then, it would seem, we leave something out when we extract the assertion from a proposition, which cannot be put back in again by simply adding the variable. The propositional function, on the other hand, already contains the variable(s): it already has, we can say, the requisite structure.

Why cannot the assertion have the requisite structure? In the passage just quoted Russell seems to just assume that there must be "no trace" of the omitted term(s), but this is presumably because he thought that the assertion must be independent of any particular term if it is to be assertible of any other term. A propositional function, on the other hand, it would seem, does indeed have some kind of dependence on the omitted term(s) through the variable(s) it contains, and hence must be taken as structured. In any case, were the assertion to have a structure, this structure could presumably only be specified by identifying the corresponding propositional function. So propositional functions would still seem to be more fundamental than assertions.

Russell also considers the case of relational propositions (pp. 85-8) and once again finds difficulties in trying to define propositional functions in terms of assertions. His conclusion is that propositional functions must indeed be taken as indefinable – as "ultimate data" (p. 88). He ends the chapter, however, by noting a further problem, which highlights just how deep the influence is at this time of a crude decompositional conception of analysis, despite the emergence of the notion of a propositional function. He writes that "according to the theory of propositional functions here advocated, the φ in φx is not a separate and distinguishable entity: it lives in the propositions of the form φx, and cannot survive analysis" (p. 88).

What does Russell mean by 'living' in a proposition? And why is what he calls 'the φ in φx' – which he also terms 'the functional part of a propositional function' (ibid.) – not an entity? It is not an entity, according to Russell, because if it were, then there would be a proposition asserting it of itself, and hence a proposition denying it of itself, giving rise to a version of the contradiction that came to bear his name (ibid.). But if it is not an entity, on Russell's view, then it cannot be a *constituent* of a proposition, and this in turn means that we cannot speak *about* it (since this would be treating it as the subject of a proposition). So there is no option but to use metaphorical expressions, talking of its 'living' in a proposition, for example. As soon as we attempt to 'cut it out', as it were, we destroy its essential nature.

4. SIGNIFICANCE AND SUBSEQUENT DEVELOPMENTS

As we have seen, Russell's notion of a propositional function arose in clarifying the difference between genuine propositions such as 'Socrates is mortal' and expressions such as 'x is mortal' that represent a class of propositions. At the time, Russell's basic distinction in the analysis of propositions was between subject and 'assertion' (cf. 1903, §43, p. 39). The notion of an assertion had been taken as one of the three fundamental notions (alongside those of class-membership and of *such that*) that characterize the class-calculus. But as soon as he had the notion of a propositional function, this replaced the notion of an assertion as one of these three fundamental notions.[28] Russell nevertheless considered whether 'propositional function' might be defined in terms of 'assertion', but came to the conclusion that it could not, and that 'propositional function' should be treated as indefinable.

Now an obvious question to ask here is whether the notion of a propositional function might be defined in terms of the more general notion of a function.[29] Propositional functions might be seen as simply functions from objects to propositions (understood as complexes of objects, on Russell's early view). Russell does not consider this question

directly in the *Principles*, but having noted the indefinability of both the notion of *such that* and the notion of a propositional function at the beginning of chapter 7, Russell remarks:

> When these have been admitted, the general notion of one-valued functions is easily defined. Every relation which is many-one, *i.e.* every relation for which a given referent has only one relatum, defines a function: the relatum is that function of the referent which is defined by the relation in question. (1903, p. 83)

The idea here can be easily explained. Consider the successor function, for example, which maps each number onto its successor. To say that x is the successor of y is to say it is the x such that $x = y + 1$. Here we make use of both the notion of *such that* and the notion of a propositional function, in this case, the propositional function '$x = y + 1$', which yields true propositions such as '$3 = 2 + 1$', and so on. Similarly, to talk of 'the father of y' is to talk of 'the x such that x is the father of y', where the propositional function here is 'x is the father of y'.[30] In Russell's later terminology, descriptive functions can be defined in terms of propositional functions.[31]

The view that descriptive functions are to be defined in terms of propositional functions is a view that Russell holds throughout his subsequent work. Treating propositional functions as more basic than descriptive functions, however, sheds little light on the notion of a propositional function itself, and in the period that immediately follows the completion of the *Principles*, its clarification is one of the main tasks that occupies Russell.[32] For present purposes what is of most interest is the distinction that he draws between two kinds of analysis, the first being traditional decompositional analysis of a whole into its parts (of a complex into its constituents), and the second being function-argument analysis. In a set of notes written in 1904, he remarks:

> What we want to be clear about is the twofold method of analysis of a proposition, *i.e.*, first taking the proposition as it stands and analyzing it, second taking the proposition as a special case of a *type* of propositions. Whenever we use variables, we are already necessarily concerned with a *type* of propositions. E.g. "$p \supset q$" stands for any proposition of a

> certain type. When values are assigned to p and q, we reach a particular proposition by a different road from that which would have started with those values plus implication, and have so built up the particular proposition without reference to a type. This is how functions come in. (1904, p. 118)

The first is decompositional analysis, the second function-argument analysis. But Russell describes the latter in a way that might strike anyone who is familiar with Frege's conception as peculiar. What Russell has in mind here is made more explicit towards the end of the notes: "We ought to say, I think, that there are different ways of analysing complexes, and that one way of analysis is into function and argument, which is the same as type and instance" (1904, p. 256).

Russell's idea might be explained by returning to Frege's example of (Lhc) – 'Hydrogen is lighter than carbon dioxide'. The first form of analysis proceeds by decomposing the proposition into its constituents – hydrogen, carbon dioxide and the relation represented by 'is lighter than'. The second form of analysis is exemplified by extracting the propositional function expressed by 'x is lighter than carbon dioxide' (which Frege takes to represent the relevant concept, understood as 'unsaturated'). This shows what *type* of proposition the proposition can be regarded as instantiating, namely, that type instantiated by 'Helium is lighter than carbon dioxide', 'Oxygen is lighter than carbon dioxide', and so on. It can also be regarded as instantiating other types, such as that instantiated by 'Hydrogen is lighter than helium', 'Hydrogen is lighter than oxygen', and so on (which Frege takes to involve the concept represented by 'x is heavier than hydrogen'). Propositional functions are not themselves *constituents* of the proposition, according to Russell, which is why a distinction is needed between the two forms of analysis.

Recognition of propositional functions, then, in the context of Russell's insistence that propositions are quite literally composed of – and uniquely determined by – their constituents, seems to drive Russell towards admitting that simple decompositional analysis is not the only form of analysis. A single proposition, such as (Lhc), can be taken to instantiate more than one propositional function – 'x is lighter than carbon dioxide', 'Hydrogen is lighter than x', 'x is lighter than y', 'Hydrogen is R-related to carbon dioxide', and so on. Since it seems implausible to suppose that *all* these propositional functions are constituents of the proposition, the identification of them cannot count as a form of decompositional analysis.

However, if it is Russell's recognition of propositional functions that seems to drive him to admit two forms of analysis, and propositional functions play a fundamental role in his philosophy from 1902 onwards, then one would expect to find him reiterating the distinction, and even clarifying it further, in his later work. But to my knowledge, Russell nowhere does so. As noted in §1 above, he distinguishes between 'material' and 'formal' analysis in his 1913 'Theory of Judgement' manuscript, but this is a distinction that is drawn under his general definition of 'analysis' as "the discovery of the constituents and the manner of combination of a given complex" (1913, p. 119).

There is an obvious answer as to why Russell did not draw the distinction later. Russell believed that everything had an ultimate analysis, and it is at this level that decompositional analysis comes to the fore. While (Lhc) permits a number of possible function-argument analyses, only one of these is arguably basic. If we take its ultimate analysis (let us say) to involve decomposition into hydrogen, carbon dioxide and the relation represented by 'is lighter than', then the most fundamental form would seem to be 'xRy', the other forms being derivable from this together with the constituents. Russell continues to deny that propositional forms are themselves constituents, but he does come to regard them as required in the ultimate (material and formal) analysis of a proposition – at any rate by 1913, when the 'Theory of Judgement' manuscript is written.[33]

If this is right, then we can say that Russell comes to see function-argument analysis as subordinate to the general process of identifying the ultimate constituents and forms of things. It is not a different "way of analysing complexes", to use Russell's 1904 phrase, on a par with decompositional analysis. It is employed in the service of decompositional analysis. In the terminology I introduced in §1 above, function-argument analysis might be better seen as a form of interpretive analysis, interpreting a given proposition as instantiating one of a number of different types (propositional forms). For Russell, this is required for ultimate decompositional analysis, but is not an end in itself.[34]

As mentioned above, the role of interpretive analysis comes out in an especially clear way in the theory of descriptions. Function-argument analysis is clearly presupposed in interpreting propositions of the grammatical form 'The *F* is *G*' as having a more complex 'underlying' logical form with a quantificational structure ('There is one and only one *F*, and whatever is *F* is *G*'). Perhaps the theory of descriptions allowed Russell to see more clearly how function-argument analysis plays a role in uncovering the 'real' logical form of a proposition, though it remains the case that he never conceptualized it in this way. Even after the theory of descriptions was firmly established as a paradigm of analytic philosophy, Russell's official conception of analysis remained staunchly decompositional. The role that the theory of descriptions played in the development of Russell's conception of analysis, however, is part of a much longer story.[35, 36]

Notes

[1] See especially Weitz 1944; Hager 1994, 2003. Weitz claims that the "fundamental element" in Russell's philosophy is his method of analysis, though he sees this method as being exemplified "in four rather distinct ways" (p. 57), which he calls 'ontological analysis' (identifying the ultimate constituents of reality), 'formal analysis' (identifying logical forms), 'logistic' (reducing mathematics to logic) and 'constructionism' (also called 'the resolution of incomplete symbols'). He regards these four ways as being united, however, by a single conception: "By analysis Russell—although he has never systematically said so—means mainly a form of definition, either real definition of a non-Aristotelian sort, or contextual definition, i.e., definition of symbols in use" (ibid.; cf. p. 110). In discussing ontological analysis in his long (23-page) first section, Weitz is clearly sensitive to many of the changes that Russell's ontological views go through. It is therefore surprising that he does not consider whether there might have been methodological changes as well. The changes in Russell's ontological views, he writes, "are due to a more rigorous application of his analytical method. Once the primacy of analysis is understood, it will become evident that there *is* a basic unity in his work, and that this unity revolves around his method" (p. 58). But whatever unity there may be, the conception of analysis as definition that Weitz identifies does not do justice to it. For one thing, the idea of contextual definition only emerges with the theory of descriptions, so in what sense was Russell an 'analytic' philosopher before 1905? If real definition is what is important before 1905, then what role does this play after 1905, and what is its relationship to contextual definition? Weitz attributes to Russell a *disjunctive* conception of analysis, but the connection between the two disjuncts needs elucidation. Weitz offers some brief remarks on the connection (see e.g. p. 116, fn. 191; pp. 119-20), but more is needed to establish that there is a *unity* of method. Nevertheless, I think Weitz has correctly identified two central elements of Russell's method of analysis, corresponding to what I call 'decompositional' and 'interpretive' (or 'transformative') analysis, as I hope this paper makes clear.

Hager (1994, 2003) has also emphasized the unity and continuity of Russell's philosophy, but sees it as grounded in a quite different conception of analysis, which he calls Russell's "two-directional view of philosophical analysis" (2003, p. 311):

> Throughout his career Russell adhered to a characteristic view of the nature of philosophical analysis according to which it has two parts. Firstly, philosophical analysis proceeds backwards from a body of knowledge to its premisses, and, secondly, it proceeds forwards from the premisses to a reconstruction of the original body of knowledge. Russell often called the first stage of philosophical analysis simply "analysis", in contrast to the second stage which he called "synthesis" (or, sometimes, "construction"). While the first stage was seen as being the most philosophical, both stages were nonetheless essential to philosophical analysis. (2003, p. 310)

What Hager is identifying here as the first part of Russell's method is what I call 'regressive' analysis. But if this is what forms part of Russell's method of analysis, then it is far from new: it goes all the way back to ancient Greek geometry (see Beaney 2007c, §2.2). So what has happened to the other two modes of analysis? Hager assimilates the decompositional mode to the regressive. He writes that "When characterizing his method of analysis, Russell sometimes, for convenience, uses 'premisses' in a *wider sense* to refer to concepts or ideas, as well as propositions" (2003, p. 313). Hager provides no textual evidence for this claim, however, which amounts to attributing to Russell confusion about the meaning of 'premise'. It is Hager who is confused here, in failing to distinguish the two modes of analysis. As far as interpretive analysis is concerned, Hager seems not to recognize it at all: in his 2003 paper, for example, he alludes just once to that 'paradigm' of analysis, Russell's theory of descriptions (p. 330). It is surprising, to say the least, that someone should attempt to elucidate Russell's method of analysis without discussion of the theory of descriptions.

On my view, as I explain in this section, one should distinguish three main modes of analysis – the regressive, the decompositional, and the interpretive. Hager identifies the first as constituting Russell's method of analysis, while Weitz identifies the second and third. That two scholars of Russell's work, with knowledge of his entire corpus, should offer such different interpretations of his supposedly core methodology, should make one cautious in talking of a unity here. As I see it, all three modes are present in Russell's philosophy, in a complex form that undergoes subtle change during the course of his thinking. Illustrating this by considering the development of Russell's conception of a propositional function is the aim of the present paper.

[2] Cf. Beaney 2007c, introductory paragraph.

[3] For a fuller account, on which I draw in the present section, see Beaney 2007b,c.

[4] Tr. in Hintikka & Remes 1974, p. 8.

[5] For more on the regressive conception, see Beaney 2007c, §2.

[6] Again, for more on the decompositional conception of analysis, see Beaney 2007c, §4.

[7] Central to analysis in ancient Greek geometry was the idea of taking something as a 'given' and working back from there – an idea that is reflected in algebra in representing the 'unknown' to be found by '*x*'. Cf. Beaney 2007c, §4.

[8] See e.g. Descartes, *Rules for the Direction of the Mind, in* Philosophical Writings, I, p. 51; *Discourse on Method*, in *Philosophical Writings*, I, p. 120.

[9] Russell 1899/1900, pp. 160-1; cf. 1903, p. 466. This remark is made in the context

of a critique of the idea of an 'organic unity'.

[10] For more on the differences between Frege and Russell here, which reflect their different conceptions of analysis, see Beaney 2003, §6; 2007b,§2; Levine 2002.

[11] Russell 1903, p. 141; cf. pp. 466-7; 1899, pp. 299-300; 1899/1900, pp. 16, 39, 160-1. Interestingly, the only entries on 'analysis' in the index to the *Principles* (which Russell prepared himself) are to these two passages (pp. 141, 466-7) where the doctrine that analysis is falsification is discussed. For later references to the doctrine, see 1914, pp. 156-8; 1918, pp. 178-9; 1959, p. 49.

[12] This is the manuscript that was abandoned as a result of criticisms that Wittgenstein made of the so-called multiple relation theory of judgement that Russell was developing here. For discussion of this, see Candlish 2007, ch. 3; Carey 2007; Hylton 1990, pp. 357-61; Ricketts 1996, §3; 2002, §§ 2-3; Stevens 2005, ch. 4.

[13] Russell 1984, p. 119. The distinction between 'material' and 'formal' analysis is reflected in Weitz's distinction between 'ontological' and 'formal' analysis; cf. n. 1 above.

[14] The problem has been much discussed in the secondary literature. See, e.g., Griffin 1993; Candlish 2007, ch. 3; Stevens 2005.

[15] This is not the first draft of material that eventually appeared in the final text. Russell had been working on what he called his 'big book' since 1897. For details of the history of the composition of the *Principles*, see Gregory Moore's editorial introductions in Russell 1993, pp. xiii-xlviii, 3-12, 181-4, 209-10. The drafts considered in what follows are all published in this volume.

[16] Russell 1899/1900, p. 39. The remark in this first draft is the same as in the published text, but with the qualification "except where what is in question is a mere collection not taken as a whole" being added after "the whole truth".

[17] The change is noted in the editorial material in Russell 1993, pp. 716, 781.

[18] See Russell 1902, pp. 211-12. The list of chapters of Part I that Russell gives corresponds exactly to what was published in the *Principles*, with the sole exception of chapter 7, the title of which is changed from 'Assertions' to 'Propositional Functions' (on which I comment shortly). The introduction of the notion of a propositional function thus seems to have been the last significant development in the evolution of the main text of the *Principles*.

[19] The whole manuscript was finished on 23 May 1902, and submitted to Cambridge University Press four days later, though small changes were made and the two appendices added during the course of the rest of the year. Cf. Moore 1993b, pp. xxxv-xxxvi.

[20] Cf. Blackwell 1985, p. 278.

[21] The issue had been much debated in the second half of the nineteenth century. Frege, too, had argued for the primacy of the propositional calculus. Cf. Beaney 1996, pp. 44-5.

[22] An obvious suggestion to make at this point is that Russell's use of the term 'propositional function' was inspired by his reading of Frege's works, which took place around this time. According to Frege, concepts are functions, concept expressions being formed by removing a proper name from a sentence, with the gap marked in some way ('( ) is a man' or 'x is a man'), in just the same way that propositional functions are formed, according to Russell. Frege may see the values of such functions as truth-values, while Russell sees the values of propositional functions as propositions, but the basic idea is similar. In fact, however, although he was aware of the significance of Frege's work from August 1900, when he first met Peano, Russell seems not to have read Frege properly until *after* he sent the manuscript of the *Principles* to Cambridge University Press in May 1902. As the passage just quoted from the *Principles* suggests, Peano had a far greater influence on Russell from August 1900 to May 1902 than Frege did.

[23] 1901, pp. 203-8. This had been chapter 7 in his 1901 plan (cf. 1901, p. 184).

[24] This is true, at any rate, in comparing the final text with what has been preserved of the 1901 draft. The final page or so of the draft, which presumably formed the basis of the second paragraph of §35 and the whole of §36 of the *Principles*, is missing.

[25] Unlike the other chapters of Part I, there is no reference to his work on this chapter, in the letters he wrote at the time, which enables us to be sure of its date of composition. But the only date possible seems to be 10 May. Cf. Blackwell 1985, p. 278.

[26] In the manuscript kept in the Bertrand Russell Archives at McMaster University, we do indeed find the original title 'Assertions' crossed out and 'Propositional Functions' written above it instead. I am grateful to Ken Blackwell for showing me this manuscript.

[27] In this respect it is worth noting that Russell's original plan suggests that the notion of an assertion is seen as one of the fundamental notions of the class-calculus (cf. 1901, p. 211, line 14); in the *Principles* itself, the notion of a propositional function is identified instead (1903, p. 19).

[28] Cf. the previous note.

[29] For a full discussion of the issue, see Hylton 1993.

[30] We could also speak of x as the object that satisfies the relevant propositional function. As Russell remarks, "*Such that* is roughly equivalent to *who* or *which*, and represents the general notion of satisfying a propositional function" (1903, p. 83).

[31] Cf. Russell & Whitehead 1910-13, *30; Russell 1919, pp. 46, 180.

[32] See especially papers 3, 5, 6, 7, 8 and 9 in Russell 1994.

[33] Even as late as 1912, when *The Problems of Philosophy* is published, we find no appeal to propositional forms in the (ultimate) analysis of propositions. Strictly speaking, by 1912, Russell has come to 'eliminate', i.e. 'analyse away', propositions themselves (understood as complexes of objects, rather than mere sentences), developing his so-called 'multiple relation theory' of judgement instead. But by 1913, under the influence of Wittgenstein, he sees that forms are needed as well, and this is the view he tries to develop in his 'Theory of Judgement' manuscript (see esp. chs. I and II of Part II). We might also note here the brief period (around 1906) in which Russell developed his 'substitutional theory', according to which propositional functions are themselves 'eliminated' by taking propositions as entities and employing the idea of substituting one entity for another within propositions. We might see this as further evidence of the dominance that the decompositional conception of analysis had on his thinking. Russell soon came round to accepting propositional functions, however, although he never held that they are constituents in propositions. For discussion of Russell's substitutional theory, see Hylton 1980, Landini 1998.

[34] This marks a fundamental difference between Russell and Frege. While Russell believed that every proposition ('content' in Frege's early terminology, 'thought' in his later terminology) had an ultimate analysis, Frege did not. For Frege, function-argument analysis was fundamental to his philosophy, underlying all of his characteristic doctrines. For discussion, see Beaney 2007d, and the works cited in n. 10 above.

[35] For discussion, see especially Hylton 2003, 2005b, 2007.

[36] A talk based on this paper was given at the conference in Riga on '200 Years of Analytical Philosophy' in August 2008, in a session on Russell with James Levine and Bernard Linsky. I am grateful to the organizers for inviting me to the conference, and to

the participants in the session, and in particular, to James and Bernard, for discussion of the issues. I would also like to thank Ken Blackwell for his help during my visit to the Bertrand Russell Archives, and David Hitchcock for his hospitality, at McMaster in October 2007.

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