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From Moore to Peano to Watson

The Mathematical Roots of Russell's Naturalism and Behaviorism

INTRODUCTION: SOME ISSUES REGARDING RUSSELL'S PHILOSOPHICAL DEVELOPMENT

Russell's philosophical development is marked by a number of key shifts in his outlook that he vividly describes in his retrospective writings. Among these are his "becom[ing] a Hegelian" in 1894 (1944a, 10; 1967, 63); his 1898 "revolt" against Idealism in which "Moore led the way, but I followed closely in his footsteps" (1959, 54); his attending the International Congress of Philosophy in Paris in August 1900 which he calls "the most important event" in "the most important year in my intellectual life" and at which he was impressed by the "precision" of Peano and his students (1944a, 12); his arriving in 1905 at his theory of descriptions, which he characterizes as his "first success" in enabling him to resolve his paradox (1959, 79); his "discover[ing] the Theory of Types" in 1906, after which "it only remained to write the book [*Principia Mathematica*] out" (1967, 152); his beginning in 1911 his association with Wittgenstein, whom he characterizes as "perhaps the most perfect example I have ever known of genius as traditionally conceived, passionate, profound, intense, and dominating" (1968, 98–9); and his "becom[ing] interested"—"during my time in prison in 1918" and influenced, at least in part, by his study at that time of the writings of the behavioral psychologist J. B. Watson—"in the problems connected

with meaning, which in earlier days I had completely ignored" (1968, 194) when "I had regarded language as 'transparent' and had never examined what makes its relation to the non–linguistic world" (1959, 145).

Until relatively recently (in particular, before the 1990's with the publication of Hylton (1990) and Griffin (1991)), the main focus of interest in Russell's philosophy, has been, I think it is fair to say, on his views from his 1905 paper "On Denoting" through his 1918 lectures "The Philosophy of Logical Atomism"[1]—that is, on the period that includes his acceptance of his theory of descriptions, his completing, with Whitehead, *Principia Mathematica* (PM), his writing the popular book *The Problems of Philosophy* (PoP), and his active engagement with Wittgenstein that leads him to abandon his 1913 manuscript, *The Theory of Knowledge* (TK), and culminates in his 1918 lectures entitled "The Philosophy of Logical Atomism" (PLA). Such a focus does not involve distinguishing Russell's early Moore–influenced post–Idealist position from the views he accepted in the wake of the 1900 Paris Congress or considering the interplay between these two aspects of Russell's development in his 1903 book, *The Principles of Mathematics* (PoM); nor does it involve any consideration of his concerns with "the problems connected with meaning" that are reflected in such post–1918 publications as "On Propositions: What They Are and How They Mean" or *The Analysis of Mind*.

Further, given a focus on Russell's writings from 1905–1918, especially on his less technical writings over that period, it is perhaps understandable that a certain picture Russell's philosophical outlook emerges, one according to which he embraces a foundationalist epistemology along with an "Augustinian" view of language, both of which reinforce the general view that the tasks of philosophy are sharply distinguished from those of science and both of which make central use of the notion of "acquaintance". For on the foundationalist epistemology that may be found in at least some of these writings, a central task of philosophy is to show how, or whether, the beliefs that are taken for granted in ordinary life and science, such as our perceptual beliefs concerning ordinary physical objects, may be justified, given that we are acquainted with sense–data but not physical objects themselves. And on the view of language that is presented in at least some of these writings, the meaning of a word (in a fully analyzed sentence) is an entity corresponding to that word, while—in accord with his so–called "principle of acquaintance"—understanding a sentence requires being acquainted with the entities corresponding to the words in that sentence, and a central task of philosophy consists in analyzing the meanings of our sentences concerning physical objects, given that we are not acquainted with such objects.

Moreover, against the background of this understanding of Russell's philosophy, it is natural to regard some of the major figures in post–World War II analytic philosophy—including, for example, the later Wittgenstein, Quine, Austin, and Sellars—as seeking to undermine characteristic features of Russell's position. This familiar view is reflected, for example, in Richard Rorty's 1979 book *Philosophy and the Mirror of Nature*. There Rorty presents Russell, along with Husserl, as seeking, in different ways, to establish philosophy as the foundational discipline, which through its knowledge of "apodictic truths" (Russell's "logical forms", Husserl's "essences") is able to assess the standing of other disciplines. According to Rorty,

> ... the kind of philosophy which stems from Russell and Frege is, like classical Husserlian phenomenology, simply one more attempt to put philosophy in the position which Kant wished it to have—that of judging other areas of culture on the basis of a special knowledge of the "foundations" of these areas. (1979, 8)

And the "story" Rorty "want[s] to tell" (ibid., 168) is how such foundationalist aspirations of Russell and Husserl were called into question by their successors:

> [I]n the end, heretical followers of Husserl (Sartre and Heidegger) and heretical followers of Russell (Sellars and Quine) raised the same sorts of questions about the possibility of apodictic truth which Hegel raised about Kant, (ibid., 167)

thereby undermining the view of philosophy as having a preeminent, privileged status. More specifically, for Rorty, Russell's mid–century successors attacked not only his views of acquaintance and his sense-data epistemology:

> [D]oubts had often been expressed about Russell's notion of "knowledge by acquaintance" ... . These doubts only came to a head, however, in the early 1950s, with the appearance of Wittgenstein's *Philosophical Investigations*, Austin's mockery of "the ontology of the sensible manifold," and Sellars's "Empiricism and the Philosophy of Mind",

but also his views of meaning:

> The distinction between the necessary and contingent—revitalized by Russell and the Vienna Circle as the distinction between "true by virtue of meaning" and "true by virtue of experience"—had usually gone unchallenged, and had formed the least common denominator of "ideal language" and "ordinary language" analysis. However, also in the early fifties, Quine's "Two Dogmas of Empiricism" challenged this distinction, and with it the standard notion (common to Kant, Husserl, and Russell) that philosophy stood to empirical science as the study of structure to the study of content. Given Quine's doubts (buttressed by similar doubts in Wittgenstein's *Investigations*) ..., it became difficult to explain in what sense philosophy had a separate "formal" field of inquiry and thus how its results might have the desired apodictic character. (Ibid., 169)

According to Rorty, these challenges to Russell's views of acquaintance along with his views of meaning "were challenges to the views idea of a 'theory of knowledge,' and thus to philosophy itself, conceived of as a discipline which centers around such a theory" (ibid.).

Recently, there has been a growing awareness that Russell's post–1918 writings call into question the sort of picture that Rorty presents of the relation of Russell's philosophy to the views of subsequent figures such as the later Wittgenstein, Quine, and Sellars. For an examination of those writings shows that by the early 1920's Russell himself was advocating views—including an anti-foundationalist naturalized epistemology, and a behaviorist–inspired account of what is involved in understanding language—that are more typically associated with philosophers from later decades whom Rorty presents as dismantling Russell's philosophy.

Hence, Thomas Baldwin begins his 2003 paper "From Knowledge by Acquaintance to Knowledge by Causation" by writing:

> There are many familiar themes in Russell's repertoire, but his later discussions of knowledge include many insights which have received little notice. Indeed, it is often supposed that in the years after 1914, after the heroic foundational phase of analytical philosophy celebrated in countless anthologies, Russell ceased to engage in creative philosophy.... One thing I want to show here is that during these years Russell was in fact developing a new conception of epistemology, linked to a new philosophy of mind, which was so far ahead of its time that it passed by largely unappreciated. It is only now that our that our own philosophy of mind has caught up with the 'naturalisation' of the mind that Russell was teaching from 1921 onwards that we can recognise in his later writings the central themes of our current debates... . (2003, 420)

And he adds later:

> [Russell's 1918] imprisonment marks his transformation from the familiar author of *Principia Mathematica* to the unfamiliar author of *The Analysis of Mind* and his subsequent writings. The key change is a new determination to bring science into philosophy: metaphysics is to be based on physics and epistemology upon psychology, and it is this latter respect that the changes are most far–reaching. (Ibid., 439)

Similarly, in his 1996 paper "Quine and Wittgenstein: The Odd Couple", Burton Dreben writes:

> By late spring of 1918, Knowledge by Acquaintance together with The Knowing Subject—the very core of what had been (Analytic) Epistemology for Russell—disappear. For the first time the nature of language *per se* is on centre stage, and Russell seeks a naturalist, indeed physicalist and broadly behaviorist account of it and of all other so-called mental activities. (1996, 48)

Numerous passages support these claims of Baldwin and Dreben.[2] Thus, for example, in his 1924 paper "Logical Atomism", Russell writes:

> I began to think it probable that philosophy had erred in adopting heroic remedies for intellectual difficulties, and that solutions were to be found merely by greater care and accuracy. This view I have come to hold more and more strongly as time went on, and it has led me to doubt whether philosophy, as a study distinct from science and possessed of a method of its own, is anything more than an unfortunate legacy from theology. (1924a, 163)

Here Russell seems to anticipate the sort of "naturalism" reflected in Quine's remark that "I see philosophy not as an *a priori* propaedeutic or groundwork for science, but as continuous with science" (1969, 126) and also suggests the sentiment behind Quine's comment that "the student who majors in philosophy primarily for spiritual comfort is misguided and probably not a very good student anyway, since intellectual curiosity is not what moves him" (1981, 193). Further, in his 1923 paper "On Vagueness", Russell writes:

> My own belief is that most of the problems of epistemology, in so far as they are genuine, are really problems of physics and physiology; moreover, I believe that physiology is only a complicated branch of physics. The habit of treating knowledge as something mysterious and wonderful seems to me unfortunate. People do not say that a barometer "knows" when it is going to rain; but I doubt if there is any essential difference in this respect between the barometer and the meteorologist who observes it. (1923a, 154)

Here he seems not only to accept a "naturalized epistemology" consistent with Quine's view that "epistemology in its new [naturalized] setting ... is contained in natural science, as a chapter of psychology" (1969, 83) but also—in his comment regarding barometers—to anticipate Daniel Dennett's discussion of the thermostat as a kind of "intentional system" (1981, 29–33).

Likewise, Russell presents himself as seeking to develop what Dreben characterizes as "a naturalist, indeed physicalist and broadly behaviorist account" of language in such remarks as these:

> [I am] one who regards thought as merely one among natural processes, and hopes that it may be explained one day in terms of physics. ... For my part, I do not regard the problem of meaning as one requiring such special methods as are commonly called "philosophical". I believe that there is one method of acquiring knowledge, the method of science; and that all specially "philosophical" methods serve only the purpose of concealing ignorance. ... Now meaning is an observable property of observable entities, and must be amenable to scientific treatment. My object has been to endeavour to construct a theory of meaning after the model of scientific theories, not on the lines of traditional philosophy; (1920a, 90–1)

> The failure to consider language explicitly has been a cause of much that was bad in traditional philosophy. I think myself that "meaning" can only be understood if we treat language as a bodily habit, which is learnt just as we learn football or bicycling. The only satisfactory way to treat language, to my mind, is to treat it in this way, as Dr. Watson does. Indeed, I should regard the theory of language as one of the strongest points in favour of behaviorism; (1927a, 43)

> We may say that a person "understands" a word when (a) suitable circumstances make him use it, (b) the hearing of it causes suitable behavior in him. We may call these two active and passive understanding respectively. Dogs often have passive understanding of some words, but not active understanding, since they cannot use words.

> It is not necessary, in order that a man should "understand" a word, that he should "know what it means," in the sense of being able to say "this word means so–and–so." Understanding words does not consist in knowing their dictionary definitions, or in being able to specify the objects to which they are appropriate. ... Understanding language is more like understanding cricket [in a footnote, Russell here cites Watson]: it is a matter of habits, acquired in oneself and rightly presumed in others. To say that a word has a meaning is not to say that those who use the word correctly have ever thought out what the meaning is: the use of the word comes first, and the meaning is to be distilled out of it by observation and analysis. Moreover, the meaning of a word is not absolutely definite: there is always a greater or lesser degree of vagueness. The meaning is an area, like a target: it may have a bull's eye, but the outlying parts of the target are still more or less within the meaning, in a gradually diminishing degree as we travel further from the bull's eye, and the bull's eye itself grows smaller and smaller; but the bull's eye never shrinks to a point, and there is always a doubtful region, however small, surrounding it. (1921, 197–8; see also 1919b, 290)

Thus, by the early 1920's, Russell seems to be expressing views regarding language that are more typically associated with the later Wittgenstein or with Quine of the 1960's. In particular, by writing that "the use of the word comes first, and the meaning is to be distilled out of it", Russell seems to be anticipating Wittgenstein's view that "we are inclined to forget that that it is the particular use of a word only which give the word its meaning" (1958, 69) as well as Quine's view that "there is nothing in linguistic meaning beyond what is to be gleaned from overt behavior in observable circumstances" (1992, 38). Moreover, in claiming that the meaning of a word we are able to "distill" out of its use "is not absolutely definite" but rather admits of "a greater or lesser degree of vagueness", Russell appears to advocate an indeterminacy thesis of the sort that Quine articulates when he writes, for example: "When ... we turn thus toward a naturalistic view of language and a behavioral view of meaning, ... [w]e give up an assurance of determinacy." (1969, 28)

Further, by accepting the view that understanding a word is a matter of using it in "appropriate circumstances" and responding to it in "suitable" ways, Russell has rejected his "principle of acquaintance". Thus, in PoP, he held not only that understanding a sentence requires being acquainted with "the meaning"—that is, the entity which is the meaning—of each word in that sentence (PoP, 58, 104), but also that "no sentence can be made up without at least one word which denotes a universal" (ibid., 93), so that he was committed to the view no one can understand any sentence without being acquainted with at least one universal. In contrast, in the chapter entitled "Language" in his 1927 book *Philosophy*, Russell writes:

> General words such as "man" or "cat" or "triangle" are said to denote "universals", concerning which, from the time of Plato to the present day, philosophers have never ceased to debate. Whether there are universals, and, if so, in what sense, is a metaphysical question, which need not be raised in connection with the use of language. The only point about universals that needs to be raised at this point is that the correct use of general words is no evidence that a man can think about universals. It has often been supposed that, because we can use a word like "man" correctly, we must be capable of a corresponding "abstract" idea of man, but this is quite a mistake. Some reactions are appropriate to one man, some to another, but all have certain elements in common. If the word "man" produces in us the reactions which are common but no others, we may be said to understand the word "man". ... Consequently there is no need to suppose that we ever apprehend universals, although we use general words correctly. (1927a, 53–4)

Again, the similarity with Quine's position is striking. For like Russell in this passage, Quine holds that although there is a genuine metaphysical issue as to whether we should countenance universals, that issue is not settled by our ability to understand sentences involving general terms. Just as Russell writes here that "the correct use of general terms is no evidence that a man can think about"—or "apprehend"—"universals", Quine argues twenty years later that "we can use general terms, for example, predicates, without conceding them to be names of abstract entities" (1948, 12).

Thus, however much the views of such figures as the later Wittgenstein or Quine may be regarded as directed against features of Russell's position prior to 1918, they do not seem to be opposed, at least straightforwardly, to his post–1918 position. On the contrary, given the apparent similarities between Russell's later views and those of his supposed "heretical follows", the question arises as to the extent to which, far from seeking to undermine Russell's position, the later

Wittgenstein and Quine were instead positively influenced, directly or indirectly, by Russell's later writings. My concern, here, however, is not to explore that question; instead, it is to show that Russell's post–1918 turn to an explicitly naturalistic characterization of philosophy and a behaviorist characterization of language[3] is not itself a wholly radical break from his prior position, but rather has its source, at least in large part, in views he accepted following the August 1900 Paris Congress of Philosophy—something, which, if true, would help explain why, as noted above, Russell (in 1944) calls that event, rather than his initial break from Idealism in 1898, "the most important event" in "the most important year in my intellectual life".

My discussion proceeds in three main parts. First, I discuss some views that Russell accepts in his post–Idealist pre–Peano Moorean philosophy—including a foundationalist epistemology, the "Augustinian" view of language, the "principle of acquaintance", and a conception of the tasks of philosophy as clearly distinguished from those of science—views that, I have just indicated, are often associated with Russell throughout his philosophical development. In the remaining two parts, I focus on two different aspects of Russell's post–Peano views of mathematics and argue that these threaten various aspects of his overall Moorean position. In Part 2, I discuss his coming to hold that that nineteenth–century mathematicians, most notably Dedekind, Weierstrass, and Cantor, had solved all the traditional problems of the infinite and continuity and argue that Russell's later anti–foundationalism along with his "naturalism" and his view of "the scientific method in philosophy" are closely connected to this post–Peano development. In Part 3, I discuss his coming to regard the cardinal numbers as "classes of similar classes". In particular, I argue that Russell's defense of that view is not in accord with his Moorean conception of analysis but rather appeals to a notion of "vagueness" that threatens to undermine the "Augustinian" view of language" and the "principle of acquaintance"; and I argue further that it is not until he accepts his behaviorist view of meaning and understanding in his post–1918 writings that Russell can allow "vague" language to be meaningful and capable of being understood and can thereby make plausible his post–Peano practice of analysis.

In thus arguing that views that Russell comes to accept in the aftermath of the Paris Congress play a central role in his coming to accept positions that become prominent post–1918, I will be presenting a different view of the relation between Russell's post–Idealist pre–Peano Moorean philosophy and his post–Peano philosophy from that which Peter Hylton presents when he writes:

> [Russell's] fundamental doctrines were the ones that he held before he was influenced by mathematical logic [that he acquired as a result of the Paris Congress], and the chief effects of that influence were to enable (or force) him to articulate those doctrines further, to show him that they could play a role in the solution of problems which had previously seemed insoluble, and, especially, to enable him to defend those doctrines. (1990, 152-3)

For I will be arguing that far from enabling him to articulate more fully and defend the "fundamental doctrines" he came to accept immediately after breaking with Idealism, the technical views he comes to accept after the Paris Congress call into question many of those earlier Moorean "fundamental doctrines" and play a central role in leading him to embrace the naturalist and behaviorist positions of his post–1918 writings. If this is correct, then the views of the stereotyped Russell who serves as a target for the mid–century philosophers Rorty highlights are, in large part the legacy of the Moorean Russell, a philosopher whom Russell himself began to undermine as early as the latter part of 1900.

1. THE MOOREAN RUSSELL: SOME BASIC COMMITMENTS

Russell's "Moorean" period (as I use the phrase) begins with his break with Idealism towards the end of 1898 and ends at the Paris Congress of August 1900. During this period, Russell wrote, among other things, *The Philosophy of Leibniz* (PoL), an entire draft of PoM (which he re–wrote after the Paris Congress), and a number of papers reflecting his views of time, space, number, and magnitude, which reflect his general views regarding the nature of order. My purpose here is twofold. First, I discuss some basic features of Russell's Moorean metaphysics, philosophy of language, and epistemology. In particular, I highlight his metaphysical atomism (§1.1); his acceptance of an "Augustinian view" of language (§1.2) and the "principle of acquaintance" (§1.3), which are central to his early conception of analysis; and his foundationalist epistemology (§1.4), which is central to his early understanding of the distinction between philosophy and the sciences (§1.5). My second main purpose here is to discuss, in the final section (§1.6) of this Part, Russell's Moorean views of time, magnitude, and number, showing, in particular, how they exemplify the general features of his Moorean outlook that I have introduced in the preceding sections.

The basic views in metaphysics, philosophy of language, and epistemology that I here attribute to the Moorean Russell are central to the stereotyped picture of Russell throughout his philosophical development that I have discussed above;[4] indeed, as I mention below, some of Russell's clearest and best known formulations of at least some of these views appear in his post–Moorean writings. My claim here, then, is only that these views are all part of his Moorean philosophy, not that he ceases to endorse them immediately after the Paris Congress. In Parts 2 and 3, however, I seek to show that views he accepts after the Paris Congress in the philosophy of mathematics threaten various elements of his Moorean philosophy, so that questions arise as to how, or whether, he can reconcile his post–Peano views regarding mathematics with his overall Moorean philosophy.

1.1. Metaphysical Atomism

Fundamental to Russell's revolt against "absolute" or monistic idealism was his acceptance of a metaphysical atomism. On the view he rejects, the universe is an "organic unity", which may not be coherently understood as composed of parts that are simpler than the whole they constitute, in which case "analysis"—the breaking down of a whole into simpler parts—"is falsification" and the "conceptual" distinctions we make in characterizing the universe do not correspond to "real divisions" of the universe "into parts".[5] Whereas monists hold that there is a mutual dependence between a whole and its parts, according to which whatever "parts" we find in a whole will be as complex as the original whole itself, Russell holds, on the contrary, that that the being of a whole depends on the being of its parts but not vice–versa, that the parts of a whole are simpler than that whole, and that analysis is complete when we have arrived at "simple terms", entities which have no parts. Thus, for example, in a passage from his pre–Peano draft of PoM that appears in PoM itself, Russell writes:

> We are sometimes told that things are organic unities, composed of many parts expressing the whole and expressed in the whole. . . . The only kind of unity to which I can attach any precise sense—apart from the unity of the absolutely simple—is that of a whole composed of parts. But this form of unity cannot be what is called organic; for if the parts express the whole or the other parts, they must be complex, and therefore themselves contain parts; if the parts have been analyzed as far as possible, they must be simple terms, incapable of expressing anything except themselves. (1899–1900, 160-1; PoM, 466)

For Russell, all wholes, including infinite wholes, are composed of simple "ultimate constituents", so that to be a simple is to be "an ultimate constituent of the universe" (1899–1900, 51-2). Unlike monistic idealists, who regard "the Absolute" as a complex but unanalyzable whole, Russell holds that, metaphysically speaking, there is no unanalyzable complexity in the universe and that what is metaphysically ultimate is the simple.

1.2. The "Augustinian" View of Language and Analysis

Russell's metaphysical atomism and his early conception of analysis are intimately connected with his acceptance of an "Augustinian" view of language that incorporates:

(Aug) For a word to be meaningful is for there to be a single entity which that word stands for and which is thereby the meaning of that word.

In the opening section of the *Philosophical Investigations* Wittgenstein quotes a passage from Augustine and finds in it "the roots of the following idea":

> Every word has a meaning. This meaning is correlated with the word. It is the object for which the word stands. (1953, §1)

On the view of language that Wittgenstein is characterizing, what constitutes a word's having a meaning—or being meaningful —is its standing for a single entity ("the object for which the word stands"); and on this view, the entity that a word stands for is "the meaning" of that word.

Russell reflects his commitment to (Aug) in PoM, where after writing:

> [I]t must be admitted, I think, that every word occurring in a sentence must have *some* meaning; a perfectly meaningless sound could not be employed in the more or less fixed way in which language employs words [here, as elsewhere, emphasis is in the original] (PoM, 42)

he adds five pages later:

> *Words* all have meaning, in the simple sense that they are symbols which stand for something other than themselves. But a proposition, unless it happens to be linguistic, does not itself contain words: it contains the entities indicated by words. (Ibid., 47)

Thus, Russell makes a seamless transition from indicating that "every word occurring in a sentence" must be meaningful—that is, "must have *some* meaning"—to indicating that its thus having a meaning consists in its standing for an entity. It is this transition that Russell rejects by the early 1920's when he comes to hold both that for a word to be meaningful is for it to be used and responded to in appropriate ways and that "the meaning" of a word we are able to "distill" out of its use "is not absolutely definite". By doing so, he thereby holds, as against (Aug), that a word can be meaningful—if it is used in appropriate ways—without its yet succeeding in standing for, or being correlated with, a single entity that we are entitled to call "the meaning" of that word.

An example of Russell's early commitment to (Aug) occurs in his discussion of the word "and" as it occurs in statements of the form "*A* and *B* are two".[6] In his pre–Peano draft of PoM (and again in PoM itself) he writes:

> What is meant by *A and B*? Does this mean anything more then the juxtaposition of *A* with *B*? That is, does it contain

> a concept over and above that of *A* and that of *B*? Is and a separate concept, which occurs besides *A*, *B*? (1899–1900, 17; PoM, 71)

Here, Russell raises the question as to whether in "*A* and *B* are two" the word "and" serves to stand for an entity or whether the only words standing for entities in such a phrase are those replacing "*A*" and "*B*". Initially, he presents considerations against the view that in that context "and" serves to stand for an entity. He argues first that "and" cannot there stand for "a relation between *A* and *B*"; for in that case "*A and B* would then be a proposition", in which case it would be a "unified complex" and so "would be one not two". Further, he argues that if *A* and *B* are distinct, then they "*are* two, and no mediating concept seems necessary to make them two" (ibid.). Having made these arguments, he continues:

> Thus *and* would seem to be meaningless. But it is difficult to maintain this theory. To begin with, it seems rash to hold that any word is meaningless. When we use the word *and*, we do not seem to be uttering mere idle breath, but some idea seems to correspond to the word. Again some kind of combination seems to be implied by the fact that *A* and *B* are two, which is not true of either separately. When we say "*A* and *B* are yellow", we can replace the proposition by "*A* is yellow" and "*B* is yellow"; but this cannot be done for "*A* and *B* are two"; on the contrary, *A* is *one* and *B* is *one*. Thus it seems best to regard *and* as expressing a definite unique kind of combination, not a relation, and not combining *A* and *B* into a whole, which would be one. (Ibid.)

In this passage, Russell applies (Aug) twice. First, having presented considerations against the view that "and" serves to stand for an entity, he writes that "*and* would thus seem to be meaningless", thereby indicating, as (Aug) requires, that a word that fails to stand for any entity is "meaningless". Second, after presenting considerations against the view that "and" is not meaningless in those phrases—since "when we use the word *and*, we do not seem to be uttering mere idle breath" and since "and" in "*A* and *B* are two" is not eliminable in the way that it is in "*A* and *B* are yellow"—he then concludes that "it seems best to regard *and* as expressing a definite unique kind of combination". Thus he indicates, again as (Aug) requires, that if a word is meaningful, then there is a "definite" entity for which it stands (in this case a "unique kind of combination" that does not bind A and B into a whole, as would a relation).

More generally, during his Moorean period, in his 1899 paper "The Axioms of Geometry", Russell writes:

> Philosophically, a term is defined when we are told its *meaning*. . . . It will be admitted that a term cannot be usefully employed unless it means something. What it means is either complex or simple. That is to say, the meaning is either a compound of other meanings, or is itself one of those ultimate constituents out of which other meanings are built up. In the former case, the term is philosophically defined by enumerating its simple constituents. But when it is itself simple, no philosophical definition is possible. (1899a, 410)

Here Russell combines his metaphysical atomism with (Aug) to move from writing that "a term cannot be usefully employed unless it means something" to indicating that what it means is an entity, either complex or simple. Likewise in *Principia Ethica* (PE), in discussing the meaning of "good", Moore writes:

> [I]f it is not the case that 'good' denotes something simple and indefinable, only two alternatives are possible: either it is a complex . . . or else it means nothing at all and there is no such subject as Ethics, (PE, 15)

thus suggesting that if a word (here "good") is to be meaningful at all, what it means is an entity, either simple or complex.

As these passages reflect, for both Russell and Moore, "philosophical" definition involves identifying the ultimate constituents of a complex entity. Characteristically, given that his real concern is not with words but with the entities they stand for, Russell alternates between writing (as in the first three sentences of the passage from 1899a) of a "term" as a linguistic item which has a meaning and writing (as in the last two sentences of that passage) of a "term" as the non–linguistic correlate of a linguistic item (so that the term itself is either simple or complex). Thus, a *word* may be said to be definable or indefinable depending on whether what it stands for (its "meaning") is a complex or simple entity, while a (non–linguistic) *entity* may be said to be definable or indefinable depending on whether or not it is itself complex or simple.

Given this conception of meaning along with the conception of analysis as identifying the simple constituents of a complex entity as well as his view, expressed also in "The Axioms of Geometry" that "whenever a term is analyzable, philosophy should undertake the analysis" (1899a, 412), Russell is committed to a program of analysis according to which if sentence S_1 contains a word standing for a complex entity, that word should be replaced by words "enumerating [the] the simple constituents" of that complex entity. If we stipulate

(Persp) Sentence S is a perspicuous (or privileged) representation of the proposition it expresses if and only if each word in S stands for a simple (ultimate) constituent of that proposition,

then, for Russell, analysis will be complete when S_1 has been transformed into S_2, where S_2 is a perspicuous or privileged representation of the same proposition that is represented non–perspicuously by S_1. Thus, when analysis is complete, we will have arrived at a sentence that mirrors the original proposition expressed in that it will contain as many words as there are ultimate constituents of that proposition.

1.3. Understanding, the Principle of Acquaintance, and Informative Analysis

In addition to holding, in accord with (Aug), that the proposition expressed by a sentence is a complex entity whose constituents are the entities corresponding to the words in that sentence, Russell accepts the following "principle of acquaintance":

(PoA) Understanding a sentence requires being acquainted with each constituent of the proposition expressed by that sentence.

Again, the best–known passages in which Russell endorses (PoA) occur in his post–Moorean writings; nevertheless, the notion of acquaintance (if not the word "acquaintance") and (PoA) are central to Russell's

Moorean philosophy.

In the penultimate paragraph of "On Denoting", Russell writes:

> In every proposition that we can apprehend . . . , all the constituents are really entities with which we have immediate acquaintance. (1905, 427)

Likewise in his 1911 paper "Knowledge by Description and Knowledge by Acquaintance" as well as in PoP, Russell writes:

> The fundamental principal in the analysis of propositions containing descriptions is this: *Every proposition which we can understand must be composed wholly of constituents with which we are acquainted.* (1911b, 154; PoP, 58)

Given that Russell holds that each sentence expresses a proposition, whose constituents are the meanings of the words in that sentence, and that understanding a sentence requires "apprehending" the proposition it expresses, then in these passages he is committing himself to (PoA). And in PoP Russell defends this "fundamental principle" by writing:

> We must attach *some* meaning to the words *we* use, if we are to speak significantly and not utter mere noise; and the meaning we attach to our words must be something with which we are acquainted. (PoP, 58)

Thus, he indicates, not only, in accord with (Aug), that the meanings of words are entities corresponding to those words, but further, in accord with (PoA), that understanding a sentence requires being acquainted with "the meaning we attach to our words". And in TK, he applies (PoA), when he writes:

> Let us take as an illustration some very simple proposition, say "*A* precedes *B*", where *A* and *B* are particulars. In order to understand this proposition, it is . . . obviously necessary that we should know what is meant by the words which occur in it, that is to say, we must have acquaintance with *A* and *B* and with the relation of "preceding". (TK, 110–1)

For Russell, that is, for a word to be meaningful is, by (Aug), for that word to stand for a single entity, which is its meaning; then, for a sentence to be meaningful is for it to express a single proposition whose constituents are the meanings of the words in that sentence; and, by (PoA), to understand that sentence is to apprehend the proposition it expresses, which, in turn requires, being acquainted with each constituent of that proposition, that is to say, requires being acquainted with the meaning of each word in that sentence.

Russell's notion of acquaintance has its source in the "act–object" distinction that Moore emphasizes in his paper "The Refutation of Idealism",[7] where he argues generally, that in every "idea we must distinguish two elements, (1) the 'object,' or that in which one differs from another; and (2) 'consciousness,' or that which all have in common -- that which makes them sensations or mental facts" (1903b, 20). Further, for Moore, in every case in which we are thus "conscious of an object", we thereby "know" or are "directly aware" of that object. As he writes in discussing a sensation of blue:

> A sensation is, in reality, a case of 'knowing' or 'being aware of' or 'experiencing' something. . . . [T]his awareness is not merely, as we have hitherto seen it must be, itself something distinct and unique, utterly different from blue: it also has a perfectly distinct and unique relation to blue . . . . This relation is just that which we mean in every case by 'knowing'. (Ibid., 24–5)

And since, for Moore, the "object" of "awareness" is not (in general) a mental item, then to be aware of an entity is to stand in the relation of "knowing" to an entity that is not (in general)"in the mind". Hence:

> There is, therefore, no question of how we are to "get outside the circle of our own ideas and sensations." Merely to have a sensation is already to *be* outside that circle. It is to know something which is as truly and really *not* a part of *my* experience, as anything which I can ever know. (Ibid., 27)

Further, for Moore, "sensation and thought" are "both forms of consciousness or, to use a term that seems to be more in fashion just now, they are both ways of experiencing" (ibid., 7), so that "the nature of that peculiar relation which I have called 'awareness of anything' . . . is involved equally in the analysis of *every* experience—from the merest sensation to the most developed perception or reflexion" (ibid., 29).

For Moore, to have an "idea" of an entity—whether by sensation or by thought—is to be "aware of" or to "know" that entity, an entity which is not (in general) a mental item.

In PoP, Russell introduces the term "acquaintance" in Chapter IV ("Idealism"), where, like Moore in "The Refutation of Idealism", he uses the act–object distinction to argue against Berkeley's view that "*esse* is *percipi*". First, he introduces the act–object distinction by discussing an ambiguity in the notion of an "idea":

> Taking the word 'idea' in Berkeley's sense, there are two quite distinct things to be considered whenever an idea is before the mind. There is on the one hand the thing of which we are aware—say the colour of my table—and on the other hand the actual awareness itself, the mental act of apprehending the thing. The mental act is undoubtedly mental, but is there any reason to suppose that the thing apprehended is in any sense mental? . . . Berkeley's view, that obviously the colour *must* be in the mind, seems to depend for its plausibility upon confusing the thing apprehended with the act of apprehension. (PoP, 41–2)

Then, in the following paragraph, he writes:

> This question of the distinction between act and object in our apprehending of things is vitally important, since our whole power of acquiring knowledge is bound up with it. The faculty of being acquainted with things other than itself is the main characteristic of a mind. Acquaintance with objects essentially consists of a relation between the mind and something other than the mind; it is this that constitutes the mind's power of knowing things. (Ibid., 42)

Thus, like Moore, Russell indicates that the object of a mental act is (in general) an extra–mental entity which the mind, in virtue of that mental act, "knows", or, in the terminology, he adopts here, is "acquainted" with.

While "The Refutation of Idealism" was published in 1904 and PoP in 1912, it is clear that Russell assumes this view of the act–object distinction throughout his Moorean period. Thus, for example, in PoL, Russell writes:

> With Locke's definition, that an idea is the object of thought, we may agree; but we must not seek to evade the consequence that an idea is not something in the mind, (PoL, 166)

thereby indicating that insofar as an "idea is the object of thought", then it is not (in general) "in the mind". Further, in the course of criticizing Lotze's view of space, in a paper he composed prior to the Paris Congress (and delivered there), Russell writes:

> The being which belongs to the contents of our presentations is a subject upon which there exists everywhere the greatest confusion. Lotze described it as the fact of being intuited by us. ... Lotze presumably regards the mind as creative in some sense; what it intuits is supposed to acquire a kind of existence which it would not have otherwise. ... But the whole theory rests, if I am not mistaken, upon the neglect of the fundamental distinction between an idea and its object. Having neglected the notion of being, people have supposed that what does not exist is nothing. Seeing that numbers, relations, and many other entities do not exist outside the mind, they have supposed that the thoughts in which they think of these entities actually create their own objects. Every one (except for philosophers) can see the difference between a tree and my idea of a tree, but few people see the difference between the number 2 and my idea of this number. And yet the distinction is as necessary in one case as in the other. I do not think the number 2, but I think of the number 2. For if it is supposed that I think the number 2 itself, then 2 is one of my thoughts and as a result this 2 differs from the 2 which is someone else's thought. Hence it cannot be said that there is a number 2, of which various people think there will be as many 2's as there are minds. ... The objects of thought possess being, whether they are thought of or not; and it is because they are that we can think of them. Their being is not a result, but a precondition, of the fact that we think of them. But as regards the existence of an object of thought, nothing can be inferred from the fact of its being thought of, since

> the object certainly does not exist in the very thought which thinks of it. Hence, finally, no special kind of being belongs to the objects of presentation as such. (1901b, 254; see also PoL, 165-6, 1901c, 277–8, PoM 450–1)

Here again, Russell is concerned to emphasize that the "object" of an "idea"—whether it be a physical object such as a tree, which exists in space and time, or an abstract entity, such as the number 2 or a relation, which, for Russell, has "being" but not existence—is (in general) an extra–mental entity whose "being" does not depend on "its being thought of". However, by discussing the act–object distinction in the context of criticizing Lotze's views of "intuition" or of the "objects" (or "contents") of "presentation", he is also indicating, like Moore, that the "object" of an "idea" is not only (in general) extra–mental but is also an entity that we thereby "know", or (in his later terminology) are thereby "acquainted" with. According to Russell, Lotze holds that since an "object of presentation", or what we "intuit", is not extra–mental, then "intuition" or "presentation" can never give us knowledge of extra–mental reality, but gives us, at best, something whose existence is acquired by our act of "intuition"; in contrast, for Russell, since what we "intuit"—or what is an "object" of our ideas—is (in general) extra–mental, then in "intuiting" an extra–mental entity or having such an entity as the object of one of our ideas, we thereby know extra–mental reality.[8] Thus, for Russell, in "thinking of" the number 2, we thereby know or "intuit" or are acquainted with, the number 2 itself; in Moore's words, we thereby "get outside the circle of our ideas".

Not only does the Moorean Russell accept the act–object distinction that underlies the notion he comes to call "acquaintance"; he also applies that notion to our apprehension of propositions so as to indicate that he accepts (PoA). Thus, in the course of criticizing Bradley's theory of judgment, in a paper he presented in May 1900 Russell writes:

> It is commonly held that every proposition ultimately ascribes a predicate to a subject, the subject being something real while the predicate is something merely ideal or mental. Thus Mr. Bradley holds that every proposition ascribes a predicate to Reality, that all predicates are ideas, while Reality is not an idea. This doctrine appears to me vicious in both parts. On the one hand, everything that can occur in

> a proposition must be something more than a *mere* idea—it must be the object of an idea, *i.e.* an entity to which an idea is related: to this extent all terms in propositions are like Mr. Bradley's Reality. On the other hand, whatever can form part of a judgment which we make must be the *object* of one of our ideas, even if it be Mr. Bradley's Reality. Thus Reality becomes assimilated to other terms. Every term is both an entity in itself, and an object to a possible idea. But the idea and its object are as distinct in the case of so–called adjectives and relations as they are in the case of Reality. (1900b, 229)

Here Russell is arguing that those who fail to recognize the ambiguity in "idea"—and who thus fail to distinguish mental act from non–mental object of a mental act—tend not only to treat some non–mental entities as mental but also to regard non–mental entities as inaccessible to us. Thus, he regards the view that "all predicates are ideas, while Reality is not an idea" as "vicious in both parts". For Russell, to hold that "all predicates are ideas", or are "merely ideal or mental", is to fail to recognize that predicates (properties) are non–mental "objects" of our ideas; and to hold that "Reality is not an idea" and so inaccessible to us is to fail to recognize that non–mental "objects" can be objects of our ideas and so can be known by us. For Russell, that is, the failure to recognize that what is "present" or "before" the mind need not be "in" the mind leads both to the view that since a predicate is "before" the mind when we make a judgment, then it is therefore "in" the mind as well as to the view that since "Reality" is not "in" the mind, then we cannot have it "'before" the mind when we make a judgment.

Hence, in writing here that "everything that can occur in a proposition must be something more than a *mere* idea—it must be the object of an idea", Russell is emphasizing here[9] that the constituents of a proposition are not in general mental items but are rather the non–mental objects of our "ideas". And in writing further that "on the other hand, whatever can form part of a judgment which we make must be the *object* of one of our ideas, even if it be Mr. Bradley's Reality", Russell is indicating that when we make a judgment and thereby apprehend a proposition, each constituent of that proposition being an "object of one of our ideas" is an entity which we are "directly aware" of. In his

later terminology, he is indicating—as he does in the passages I have cited above from his later writings—that apprehending a proposition requires being acquainted with each of its constituents, so that, assuming that he holds that understanding a sentence requires apprehending the proposition it expresses, he is thereby committing himself to (PoA).[10]

Consistent with (PoA), Russell indicates in his Moorean period that while understanding a defined term requires understanding the indefinable terms in terms of which the original term is ultimately defined, understanding an indefinable term requires "intuitively apprehending"—or, in his later terminology, being acquainted with—the meaning of that indefinable term. Thus, for example, in "The Axioms of Geometry", he writes:

> There has at all times been a wide–spread notion that a term cannot be understood unless it is defined. This means: You cannot know what *A* means, except in terms of *B*, nor what *B* means except in terms of *C*. To this process there is evidently no end, and no one can ever know what anything means. Unless, then, some terms can be understood without a definition, no term can be understood by the help of a definition. All these points are so obvious that I should be ashamed to mention them, but for the fact that mathematicians persistently ignore them, (1899a, 411)

and adds shortly thereafter:

> [T]he meaning of the fundamental terms cannot be given, but can only be suggested. If the suggestion does not call up the right idea in the reader, there is nothing to be done. (Ibid., 412)

Likewise, in his earlier 1898 manuscript "An Analysis of Mathematical Reasoning", Russell writes:

> It is the habit of mathematicians to begin with definitions . . . and to assume that definitions, in so far as they are relevant, are always possible. It is, however, sufficiently evident that some conceptions, at least, must be indefinable. For a conception can only be defined in terms of other conceptions, and this process, if it is not to be a vicious circle,

> must end somewhere. In order that it may be possible to use a conception thus left undefined, the conception must carry an unanalyzable and intuitively apprehended meaning. Intuitive apprehension is necessary to the student, since he is otherwise unable to understand what is meant.... All that can be said is, that whoever is destitute of this apprehension cannot successfully study the subject in hand, and that any attempt to give him, by means of formal definitions, the conceptions in which he is lacking, is a fundamental error in Logic. (1898, 163)

In these passages, Russell indicates, in accord with (PoA), that our understanding of indefinable terms is a necessary condition for understanding any definable term and further that our understanding of an indefinable term is simply a matter of having an "idea" of, or "intuitively apprehending", the simple entity that is the meaning of that term. Moore expresses the same view in *Principia Ethica* when he writes:

> My point is that 'good' is a simple notion, just as 'yellow' is a simple notion; that just as you cannot, by any manner of means, explain to any one who does not already know it, what yellow is, so you cannot explain what good is. Definitions of the kind that I was asking for, definitions which describe the real nature of the object or notion denoted by a word ... are only possible when the object or notion in question is something complex. ... [W]hen you have enumerated them all [all the simple parts of something complex], when you have reduced [the complex item to its] simplest terms, then you can no longer define those terms. They are simply something which you think of or perceive, and to any one who cannot think of or perceive them, you can never, by any definition make their nature known. (PE, 7)

And Russell continues to hold this view in 1913, when he writes: "[E]very series of definitions ... must have a beginning, and therefore there must be undefined terms .... The undefined terms are understood by means of acquaintance." (TK, 158) On the view expressed in all these passages, and in accord with (PoA), while understanding an undefined term requires being acquainted with he simple entity designated by that term, understanding a defined term requires being acquainted with the complex entity it designates; for understanding the defined term requires understanding the undefined terms used in its definition, and hence requires being acquainted with the simples corresponding to those undefined terms.

Accepting (PoA) requires denying that analysis can be "illuminating" in the sense of revealing any constituents of a proposition with which one was not already acquainted in apprehending that proposition prior to analysis. For, by (PoA), understanding a sentence—regardless of whether or not it is a perspicuous representation—requires being acquainted with the constituents of the proposition expressed by that sentence. Hence, by (PoA), even if S_1 is a non–perspicuous representation of the same proposition that S_2 expresses perspicuously, one who understands S_1 is acquainted with exactly the same simple entities that are designated by the words in the perspicuous representation S_2. While the analysis can make explicit the entities with which one is acquainted in understanding a non–perspicuous representation in that it will contain a separate word for each ultimate constituent of the proposition expressed, it cannot reveal any entities with which one was not already acquainted in understanding a non–perspicuous representation of that proposition.

Thus, if in understanding sentences S_1 and S_2 one has different entities "before the mind", then applying (PoA) requires one to conclude that those sentences express different propositions. And in *Principia Ethica*, Moore, in effect, so applies (PoA) when he writes:

> Every one does in fact understand the question 'Is this good?' When he thinks of it, his state of mind is different from what it would be, were he asked 'Is this pleasant, or desired, or approved?' It is has distinct meaning for him .... Whenever he thinks of 'intrinsic value,' or 'intrinsic worth,' or says that a thing 'ought to exist,' he has before his mind the unique object —the unique property of things—which I mean by 'good'. Everybody is constantly aware of this notion.... (PE, 16–17)

Here, he argues, in accord with (PoA), that since when we understand the questions "Is this good?" and "Is this pleasant?" (or "Is this desired?” or “Is this approved?”), we have different entities “before the mind”, then these sentences express different propositions. In his 1909 paper “Pragmatism”, Russell argues similarly against pragmatist accounts of “the meaning of truth”. Thus, after writing generally:

> When we ask ‘What does such and such a word mean?’ what we want to know is ‘What is in the mind of a person using the word?’

he adds in the following paragraph:

> When we say that a belief is true, the thought we wish to convey is not the same thought as when we say that the belief furthers our purposes; thus ‘true’ does not mean ‘furthering our purposes’.... Thus pragmatism does not answer the question: What is in our minds when we judge that a certain belief is true? (1909, 274)

For Russell, since what is “present” to our minds,[11] when we understand “That belief is true” and “That belief furthers our purposes”, is not the same, then, in accord with (PoA), these sentences express different propositions.

In these passages, Moore and Russell are, in effect, applying (PoA), against proposed analyses of central concepts and against claims that given sentences express the same proposition; and it may seem that accepting (PoA) precludes one from ever defending any philosophically interesting analyses. For such an analysis would require one to recognize a case in which sentence S_2 provides a perspicuous representation of the same proposition that is expressed non–perspicuously by a sentence S_1; but if to be “philosophically interesting”, an analysis would have to reveal something which was not “present to the mind” prior to the analysis, then any philosophically interesting analysis would conflict with (PoA). In fact, however, I argue below (§1.6), that for the Moorean Russell, there are philosophically significant analyses that require us to distinguish perspicuous from non–perspicuous representations of the same propositions but that are not in conflict with (PoA), and so do not reveal anything that someone who understands the non–perspicuous representation does not already recognize. In contrast, as I argue in Part 3, the post–Peano Russell defends analyses in the philosophy of mathematics—perhaps, most notably the analyses involved in his

definitions of the cardinal numbers—that he never presents as simply reflecting what is “present to our minds” in understanding the relevant words prior to those analyses. Hence, these post–Peano analyses conflict with (PoA) and raise a problem for Russell that I argue he does not resolve until his post–1918 writings in which he adopts a behaviorist account of what is involved in understanding language.

1.4. Epistemological Foundationalism

Besides holding that that all definition depends ultimately on indefinable terms whose meaning can be known only by “intuitive apprehension” or acquaintance, the Moorean Russell also accepts a foundationalist epistemology according to which all proof or justification depends ultimately on “indemonstrable” propositions—namely, “axioms” or “ultimate premisses”—whose truth is “self–evident” or “intuitively apprehended”. In fact, both after and during his Moorean period, Russell indicates that just as we avoid vicious regress in definition by recognizing indefinable terms whose meaning we know by acquaintance, so too we avoid vicious regress of justification by recognizing “self–evident” propositions, propositions whose truth cannot be justified by any other propositions taken to be true.

Thus, in his 1913 manuscript *Theory of Knowledge*, in a passage part of which I have quoted above, Russell writes:

> The vulgar imagine that, in a science, every term ought to be defined and every proposition ought to be proved. But since human capacity is finite, what is known of a science cannot contain more than a finite number of definitions and propositions. It follows that every series of definitions and propositions must have a beginning, and therefore there must be undefined terms and unproved propositions. The undefined terms are understood by means of acquaintance. The unproved propositions are known by means of self–evidence. (TK, 158)

And as early as the “Introduction” to his 1898 manuscript “An Analysis of Mathematical Reasoning”—also in a passage part of which I quoted in the previous section—Russell writes:

> It is the purpose of the present work to discover those conceptions, and those judgments, which are necessarily presupposed in pure mathematics. It is the habit of mathematicians ... to assume that definitions, in so far as they are relevant, are always possible. It is, however, sufficiently evident that some conceptions, at least, must be indefinable. ... But besides the fundamental conceptions, we must have fundamental judgments, or axioms, which form the rules of inference—or, in a certain formal sense, the major premisses—of arguments which use the fundamental concepts. ... In a science which can be exhibited as deductive, it seems evident that some such judgments are necessary; and indeed the necessity of axioms has been recognized more freely than that for indefinable conceptions. Such fundamental judgments, or axioms, will be found wherever we have two or more fundamental conceptions. Their truth must, for a successful study, be intuitively apprehended; but it must not be supposed that their truth depends upon such apprehension. On the contrary, if they are truly fundamental, no reason whatever can be given for their truth. (1898, 163)

Thus, the twofold task that Russell sets for himself here is, first, to identify the "indefinable" terms of mathematics—that is, "those conceptions ... which are necessarily presupposed in pure mathematics"—and the unprovable propositions of mathematics—that is, the "fundamental judgments" or "axioms" that "are necessarily presupposed in pure mathematics" but are such that "no reason whatever can be given for their truth". For Russell, just as there can be no definitions at all unless some terms admit no definition in simpler terms, so too there can be no justification or proof of any proposition unless some propositions are accepted as true without any proof or justification. And for Russell, just as understanding an indefinable term is independent of our understanding any other term and is instead a matter of "intuitively apprehending" the meaning of that term, so too recognizing the truth of an "fundamental" proposition or "axiom" is independent of our knowledge of any other truth and is instead a matter of "intuitively apprehending" the truth of that proposition.

Russell labels as "self–evident" a "fundamental" proposition which cannot be justified by any other proposition and whose truth can only be "intuitively apprehended", since the only "evidence" for that proposition is that not any other proposition, but only that proposition itself. Thus, in PoL, he writes of "ultimate premisses" as propositions which "have no evidence except self–evidence" (166). And Moore writes in *Principia Ethica*:

> The expression 'self–evident' means properly that the proposition so called is evident or true, *by itself* alone; that it is not an inference from some proposition other than *itself*. ... [W]hen any proposition is self–evident, ... there are no reasons which prove its truth. (PE, 143–4)

For Moore, as for Russell, where a proposition cannot be justified by any other proposition, then "no reason can be given for its truth". For on their view, to justify a proposition or to prove that it is true is to cite another proposition we believe from which the truth of the original proposition in question follows. Hence, to say that a proposition is "self–evident" is not to say that there is any "evidence" on the basis of which we infer that that proposition is true; rather, it is to say that we recognize the truth of that proposition without any ratiocination in an act of immediate insight.

1.5. The Distinction between Philosophy and Science and the Method of Philosophy

Not only does the Moorean Russell hold that there must be indefinable terms and unprovable propositions; he also holds that the characteristic philosophical activity consists of "intuitively apprehending" the meaning of indefinable terms—or, given his non–linguistic notion of "term", "intuiting indefinables"—and "intuitively apprehending" the truth of indemonstrable propositions.

Thus, as I have just indicated, the two tasks he sets for himself in his 1898 manuscript "An Analysis of Mathematical Reasoning" are to "discover" the indefinable "conceptions" and indemonstrable "axioms" of mathematics. More generally, in his book on Leibniz, he writes:

> [T]he business of philosophy is just the discovery of those simple notions, and those primitive axioms, upon which any

> calculus or science must be based. ...[A]n emphasis on *results* rather than premisses . . . is radically opposed to the true philosophic method. . . . [T]he problems of philosophy should be anterior to deduction. An idea which can be defined, or a proposition which can be proved, is of only subordinate philosophical interest. The emphasis should be laid on the indefinable and indemonstrable, and here no method is available save intuition. (PoL, 170-1)

And in PoM, he expresses the same sort of view when he writes:

> The distinction of philosophy and mathematics is broadly one of point of view: mathematics is constructive and deductive, philosophy is critical, and in a certain impersonal sense controversial. Wherever we have deductive reasoning, we have mathematics; but the principles of deduction, the recognition of indefinable entities, and the distinguishing between such entities, are the business of philosophy. Philosophy is, in fact, mainly a question of insight and perception. . . . A certain body of indefinable entities and indemonstrable propositions must form the starting-point for any mathematical reasoning; and it is this starting-point that concerns the philosopher. When the philosopher's work has been perfectly accomplished, its results can be wholly embodied in premisses from which deduction may proceed. .... All depends, in the end, upon immediate perception; and philosophical argument, strictly speaking, consists mainly of an endeavour to cause the reader to perceive what has been perceived by the author. The argument, in short, is not of the nature of proof, but of exhortation. (PoM, 129-30)

On the conception of philosophy Russell expresses in these passages, philosophy is clearly distinguished from science and is a discipline that essentially involves neither argument nor co–operation with others.

First of all, given his foundationalist view of definition and justification, Russell is in a position to distinguish sharply the "constructive and deductive" tasks of the mathematician and scientist from the "critical" concerns of the philosopher. For Russell, the mathematician or scientist

is concerned with deducing truths that have not previously been recognized, a task that may be facilitated by defining, or "constructing", concepts that have not been previously been defined; in contrast, the philosopher's concern is with the "starting–points" for any definition or deduction. Thus, for Russell, the scientist and philosopher proceed in opposite directions in the chains of definition and deduction: the scientist moves forward to deduce more truths and construct more definitions; the philosopher moves backward to identify the "indefinable" and the "indemonstrable" which makes scientific practice possible.

Further, since, for Russell, the indefinable and indemonstrable can be recognized only by "intuition" or "immediate perception" (or, in his later terminology, "acquaintance")—and hence not by any inference—the philosopher's concerns lie precisely in those areas where it is not possible to provide reasons for one's position and hence where no argument is possible. Thus, for Russell, the proper method of philosophy does not essentially involve argument or reason–giving; instead, it involves "intuition" or "immediate perception". And since it is ultimately up to each individual to intuit what he or she intuits, philosophy becomes "in a certain impersonal sense controversial". For if you fail to intuit what I have intuited, there is no rationally compelling argument by which I can convince you that what I have intuited is really there; the most I can try to do is to "cause" you "to perceive" what I have "perceived", so that what is involved here is not rational argument but rather "exhortation". If my "exhortation" fails, we are left—"in a certain impersonal sense"—with a dispute that cannot be not rationally adjudicated.

Accordingly, in *Principia Ethica*, in which he holds not only that "good" is indefinable but also that propositions as to what is good in itself are indemonstrable, Moore emphasizes that he does not regard himself as providing arguments for his view as to which propositions of the form "X is good in itself" are true. As he writes:

> [F]or answers to the . . . question [What is good in itself?], no relevant evidence whatever can be adduced: from no other truth, except themselves alone, can it be inferred that they are either true or false. We can guard against error only by taking care, that, when we try to answer a question of this kind, we have before our minds that question only,

and not some other or others. . . . (PE, viii)

For Moore, to determine whether a proposition of the form "X is good" is true, all I can do is to try to make sure that I have that proposition alone and not any other "before my mind". But once that proposition is "before my mind", I will either "intuit" that it is true or I will not; and if I fail to "intuit" the truth of such a proposition that Moore "intuits" as true, there are, Moore recognizes, no reasons for his view and against mine and hence no arguments that either of us can produce that will provide a good reason for a change in view of the other.

While this conception of philosophy might seem to hold out little hope for ending philosophical conflict or for definitively addressing issues of ultimate metaphysics, both Russell (during this period) and Moore are confident that, despite incorporating the view that philosophy concerns areas where there can be no reasons for one's position, their conception of philosophy and philosophical method is, in fact, capable, of producing not only agreement but also results regarding the ultimate constituents of the universe. For on their view, there is no in–principle impediment to our accessing those ultimate constituents—no Kantian conditions "in us" that prevent our knowing the world as it is "in itself". Moreover, on their view, "intuitively apprehending" the ultimate constituents of the universe involves no more, and no less, than attending to "what we mean" or "what is present to the mind" when we understand the sentences we utter. Accordingly, for Russell (during this period) and Moore, what stands in the way of philosophical progress is not the inability of some philosophers to understand the correct but subtle arguments of other philosophers but rather the subtle but incorrect arguments of some philosophers that have prevented philosophers generally from simply attending to the meanings of our words, meanings that we are all aware of—or are acquainted with—when we understand our ordinary discourse. Precisely because philosophy concerns areas where the proper method is "intuition", not rational argument, philosophy has been led astray by arguments that have prevented philosophers from attending to what is "before our minds".

Thus, in *Principia Ethica*, Moore claims that if we practice the "method of absolute isolation" and are careful to isolate propositions of the form "X is good in itself" from all other propositions, then "the question we have to answer"—namely, "What things have intrinsic value and in what

degrees?"—"is far less difficult than the controversies of Ethics might have led us to expect" (PE, 188). For Moore, since we are all "constantly aware of" the "unique object . . . which I mean by 'good'", if we all practice the "method of absolute isolation", we will agree as to which things are good in themselves and to what degree. Thus, for Moore: "[O]nce the meaning of the question is clearly understood, the answer to it, in its main outlines, appears to be so obvious, that it runs the risk of seeming to be a platitude." (Ibid.) For Moore, what stands in the way of agreement on fundamental issues of ethics is not the appreciation of subtle argument, but rather the inability focus solely on the question at issue.

Likewise, for the Moorean Russell, bad arguments against the reality of abstract entities—including numbers, relations, and properties—have prevented philosophers from recognizing what we will all recognize if we simply attend to what we think, including, for example, that just as a tree is different from our idea of a tree, so too the number 2 is different from our idea of the number 2. Like Moore, who holds that since "every one does in fact understand the question 'Is this good?'", we are all "constantly aware of" what "I mean by 'good'", Russell holds that when we think of the number 2, we are all aware of the same object. For Russell as for Moore, if we simply attend to what we are aware of, we will readily acknowledge abstract entities, such as "intrinsic value" or the number 2, notwithstanding philosophers' arguments to the contrary.

1.6. The Moorean Russell on Time, Magnitude, and Number

During his Moorean period, one of Russell's main concerns is the nature of order. He distinguishes generally between absolute and relative theories of order and applies that distinction to theories of time, magnitude, and number, as well as space (which, is more complicated since it involves more than one dimension), colors, and pitches of sounds. The way he addresses this topic exemplifies a number of central features of his philosophy during this period that I have emphasized above.

In "Is Position in Time Absolute or Relative?", a paper he presented in May 1900, Russell introduces the distinction between absolute and relative theories of time as follows:

> Does an event occur at a time, or does it merely occur before certain events, simultaneously with others, and after a third set? The relational theory of time holds the latter view: it holds, that is to say, that events acquire position in the time–series solely by their mutual relations, and not by relations to moments at which they occur. The absolute theory, on the contrary, holds that events occur *at* times, that times are before or after each other, and that events are simultaneous or successive according as they occur at the same or different times. (1900b, 222)

As Russell presents them, the relational and absolute theories of time countenance different indefinables. On the relational theory, the only indefinable terms to be related are events, so that "times do not really exist" (1901b, 242). While "we can say, if we wish" that a given moment is "constituted" by the "whole" of simultaneously occurring events (ibid., 243; see also 1900b, 226–7), in doing so we are treating times as definable and so not among the ultimate constituents of the universe. Further, on this theory, given any two events e_1 and e_2, one of three cases will obtain: either e_1 is before e_2, or e_1 is after e_2, or e_1 is simultaneous with e_2. So in addition to events, the relative theory of time recognizes three primitive relations: the asymmetric transitive relations of *before* and *after*, and the symmetric, transitive relation of *simultaneity*.

On the absolute theory, in contrast, there are both events and absolute moments among the ultimate constituents of the universe. Here, moments have an "intrinsic order" to one another, while events acquire a temporal order only "by correlation" with the "independent" or "self-sufficient" series of moments in absolute time (see 1901a, 291). Between any two moments, m_1 and m_2, only two cases are possible: either m_1 is before m_2, or m_1 is after m_2. For since moments *are* temporal positions, distinct moments are distinct temporal positions, in which case, no two moments can be simultaneous. In addition to the relations of *before* and *after*, now understood as relations between moments not events, the absolute theory of time recognizes a further (many–one) relation of *occurring at* which relates each event to the moment at which it occurs.

For Russell, one way to focus the difference between the two theories is to consider the analysis of the proposition expressed by an instance of

(Time$_1$) Event α is simultaneous with event β .

In accord with (Aug), Russell holds that that the expression "is simultaneous with" has as its meaning an entity—a relation, namely *simultaneity*; however, the philosophical issue for Russell is whether that relation is an indefinable, an ultimate constituent of the universe, or whether it is definable. On the relative theory of time, that relation is indefinable, so that an instance of (Time$_1$), expresses a proposition that has three ultimate constituents—namely, the events in question and *simultaneity*. In contrast, on the absolute theory, *simultaneity* is to be analyzed in terms of *occurring at* the same moment, so that the full analysis of a proposition expressed by an instance of (Time$_1$) is given by the corresponding instance of

(Time$_2$) There is a moment t such that α occurs at t and β occurs at t.

As Russell writes, on the absolute theory

> "*A* is simultaneous with *B*" requires analysis into "*A* and *B* are both at one time". (1899–1900, 147)

Hence, to accept the absolute theory of time is to hold that while corresponding instances of (Time$_1$) and (Time$_2$) express the same proposition, the instance of (Time$_2$), but not (Time$_1$), is a perspicuous representation of that proposition.

Likewise, for Russell, the central issue distinguishing the relative from the absolute theory of magnitude is whether, when two quantities are equal in magnitude, there is some further indefinable entity—a magnitude—that is common to the two quantities. On the relative theory of magnitude, there is no such indefinable magnitude and the (transitive, symmetrical) relation of *equality in magnitude* is an indefinable relation between quantities, so that an instance of

(Mag$_1$) Quantity α is equal in magnitude to quantity β

expresses a proposition that has three ultimate constituents—the quantities α and β and the indefinable relation of *equality in magnitude*. In contrast, on the absolute theory of magnitude, the relation of *equality of magnitude* is definable and an instance of

(Mag$_2$) There is a magnitude m such that α has m and β has m

is the privileged representation of the proposition expressed by the corresponding instance of (Mag$_1$). As Russell writes in his pre–Peano 1899–1900 draft of PoM:

> The kernel of the difference between the present [relative] theory and the former [absolute theory] is, that now equality is taken as indefinable, whereas formerly each magnitude was indefinable. ... The present theory is simpler than the former, since it does not require so many indefinables. Equal, greater and less are now all the apparatus of ultimate notions, whereas formerly every possible magnitude formed part of this apparatus. It might perhaps be thought, by those who regard definition as subject to convenience, that the present theory is not incompatible with the former. This would, however, be a grave philosophical error. Every concept is necessarily either simple or complex, and it is not in our power to alter its nature in this respect. If it is complex, it should be analyzed and defined; if simple, it should be used in defining other terms, without itself receiving a definition. Thus equality either may be analyzed into sameness of magnitude, or it may not be so analyzed. ... It does not lie with us to choose what terms are to be indefinable; on the contrary, it is the business of philosophy to discover these terms. We have to decide whether the indefinable term is the relation of equality, or a common property of equal quantities. If we choose the former alternative, we shall have to deny a common property; for if there were any common property, this could be used to define equality. (1899–1900, 57–8)

Thus, in accord with his overall Moorean philosophy, Russell here presents the choice between these theories as the sort of issue that is "the business of philosophy" to address. In deciding which of these theories is correct, we are not concerned with matters of convenience or with arriving at a theory which has the fewest indefinables; rather, we are attempting to determine what are among the indefinable, ultimate constituents of the universe. Are there, in addition to quantities, indefinable magnitudes? Or, are there no such magnitudes, but instead an indefinable symmetric transitive relation of *equality in magnitude*?

Again, in distinguishing the relative from absolute theory of number, Russell indicates that whereas

> [T]he relational theory would hold that there is never a number of terms at all, but there are merely the relations of equal, greater, and less among collections, [whereas on the absolute theory] equality ... consists in possession of the same number. (1900b, 225)

Thus, on the relative theory of number, there are no indefinable numbers in addition to "collections", so that the proposition expressed by an instance of

(Num$_1$) Class α is equal in number with class β

has as among its constituents an indefinable relation of *being equal in number*. In contrast, on the absolute theory of number, there are indefinable numbers but no indefinable relation of *being equal in number*, so that an instance of

(Num$_2$) There is a number n such that α possesses n and β possesses n.

is the privileged representation of the proposition expressed by the corresponding instance of (Num$_1$).

More generally, for Russell, deciding between relative and absolute theories of order requires considering instances of

(Ab$_1$) $E(\alpha, \beta)$,

where E is a symmetrical transitive relation (such as *simultaneity*, *equality in magnitude*, or *equality in number*), and

(Ab$_2$) $(\exists x)(R(\alpha, x) \,\&\, R(\beta, x))$,

where R is an appropriate many–one relation (such as *occurring at*, *having*, or *possessing*). To accept a relative theory of order is to hold that the relevant transitive, symmetrical relation is indefinable so that the relevant instances of (Ab$_1$) express propositions that contain that relation as an ultimate constituent. In contrast, to accept the corresponding absolute theory of order is to hold that that symmetrical transitive relation is definable, and that the relevant instances of (Ab_2) are privileged representations of the propositions expressed non–perspicuously by corresponding instances of (Ab_1).

During his Moorean period and even in his draft of PoM immediately following the Paris Congress, Russell accepts absolute theories of order. Moreover, in accord with his Moorean conception of the proper method of philosophy, as well with (PoA), Russell indicates that we are in a position to settle the issue by immediate "inspection" alone. Thus, in his pre–Peano draft of PoM, Russell defends the absolute theory of magnitude by writing: "[W]hen we consider what we mean when we say that two quantities are equal, it seems preposterous to maintain that they have no common property not shared by unequal quantities" (58; see also PoM 164). In accord with (PoA), that is, Russell claims that if we simply consider "what we mean" when we utter a sentence of the form (Mag_1), we will recognize that the proposition expressed does not have an indefinable relation of *equality in magnitude* as an ultimate constituent but rather that that proposition is perspicuously represented by the corresponding instance of (Mag_2). Thus while Russell is not taking himself to reveal anything that someone who understands a sentence of the form (Mag_1) does not already recognize, he is taking himself to be making a philosophically significant point, which establishes the absolute theory of magnitude, according to which there are indefinable magnitudes among the ultimate constituents of the universe. In accord with his overall Moorean outlook, Russell is indicating that questions of ultimate metaphysics can be established by immediate "inspection" and by a conception of analysis that is in accord with (PoA).

Likewise, in defending the absolute theory of time, the Moorean Russell writes: "A direct consideration of the question . . . makes it very difficult to hold that simultaneous events have absolutely nothing in common beyond the common qualities of all events" (1900b, 227); and in defending the absolute theory of number, he writes that numerical "equality [between classes] plainly consists in possession of the same number" (ibid., 225; see also 1899–1900, 146).[12] More generally, he writes: "[F]or my part I consider it self–evident that all symmetrical transitive relations are analyzable" (1901c, 262); and in his post–Peano draft of PoM he incorporates what he thus regards as "self–evident" into an "axiom" according to which the full analysis of an instance of (Ab_1)

is given by the corresponding instance of (Ab_2):

> [M]y axiom of abstraction, which precisely stated, is as follows: "Every transitive symmetrical relation, of which there is at least one instance, is analyzable into joint possession of a new relation to a new term, the new relation being such that no term can have this relation to more than one term, but that its converse does not have this property." (See PoM, 220, as correlated with Byrd, 1996–7, 165–6)

In these passages, Russell is indicating that the metaphysics of time and number, and more generally of order, can be settled by appeal to what is obvious by "direct consideration" of what we mean by sentences of the form (Ab_1), or by a "self–evident axiom" regarding the analysis of propositions expressed by sentences of that form. And for the Moorean Russell, this appeal to the obvious or "self–evident", rather than argument, does not reflect a weakness of his position, but is rather the correct methodology when it comes to settling the fundamental matters that are "the business of philosophy".

Moreover, in defending absolute theories of order, Russell indicates, in accord with his general Moorean outlook, that what has prevented philosophers from recognizing what should be obvious to all of us by "direct inspection" are widely accepted philosophical theories, in particular a "scholastic logic", which takes the subject–predicate form as fundamental and thereby fails to recognize relations. Thus, he concludes his May 1900 defense of the absolute theory of time by writing:

> On the usual subject–predicate doctrine, it is impossible to admit such a relation as that of an event to the time at which it occurs. In this relation, both terms exist, and neither is a mere predicate of the other. But if we once admit, as I hold that we must, that relations are not essentially reducible to the subject–predicate form, such an objection vanishes. If we hold to the objection, we must also deny before and after even among events, since these relations, as we have seen, are not reducible to the predicates. Thus it seems that, at bottom, the denial of reality to everything that appears real to commons sense—a denial increasingly characteristic of idealistic systems since

> Descartes—is made in the interests of a scholastic logic, not re–examined in its fundamentals by any modern writer. . . . [It is in the theory of space and time especially that the traditional logic has wrought havoc, and it is time that contradictions should cease to be regarded as commending a theory, or that admission of the obvious should be held to condemn a philosopher.] . . . Hence the grounds for rejecting a commons sense theory of time, which rest upon a dogmatically assumed scholastic logic, appear wholly inadequate. . . . (1900b, 232–3; the bracketed sentence occurs in a manuscript draft of the paper, see CP3, 783)

For Russell, if relations are not recognized—in particular, if non–symmetrical relations, including that of *occurring at*, which is needed for the absolute theory of time, or relations of *before* and *after*, which are needed for either the absolute or relative theories, are not recognized—then no theory of order, absolute or relative, is coherent, and one is faced with holding that all order is infected with "contradiction".[13] However, once relations are admitted, then there is nothing standing in the way of accepting the "commonsense " or "obvious" theory of time, the theory that admits what "appears real to common sense". For the Moorean Russell, appealing to what is "self–evident", or to what is obvious by "inspection", or to the (PoA), which simply takes into account "what we mean" by the sentences we utter, can be philosophically significant because it stands in contrast to accepting "dogmatically assumed" philosophical theories that have prevented us from acknowledging what we all, in fact, "immediately perceive" regarding the ultimate constituents of reality.[14]

Moore himself also accepts views of propositions expressed by sentences of the form (Ab_1) that are characteristic of the Moorean Russell. Thus, in his 1901 paper "Identity", Moore denies that "exact similarity [is] an unanalysable relation" and instead is concerned "to *define* the relation of exact similarity between two things as involving a relation to a third thing" (131), where "this third thing is the Platonic idea, or, as we may now call it, the universal" (132), claiming that this is what "is meant by exact similarity" (ibid.). Thus, Moore holds that the proposition expressed by a sentence of the form

(Sim_1) α is exactly similar to β

does not contain the transitive, symmetric relation of *exact similarity* as an ultimate constituent (an unanalysable relation) and holds as well, in accord with Russell's "axiom of abstraction", that the corresponding sentence of the form

(Sim_2) $(\exists x)(R(\alpha, x)\ \&\ R(\beta, x))$,

where R is the appropriate relation, provides a privileged representation of the proposition thus expressed.

Further, as late as 1911, in lectures that were later published as *Some Main Problems of Philosophy*, Moore accepts the Moorean Russell's "absolute" theory of number, which, as I discuss in Part 3 below, Russell himself had already rejected by the publication of PoM in 1903. In particular, in discussing whether there are any universals of what he calls his "third kind"—universals that are neither relations nor relational properties—Moore writes generally:

> It may be held . . . that when we say that two things *resemble* one another, what we mean by this is always merely that they have some property in common. It may be held in short, that resemblance always consists in the possession of some common property—is merely another name for such possession. (1953, 358)

Thus, Moore is considering the view, which would follow from Russell's "axiom of abstraction", that a sentence of the form "α resembles β (in a given respect)" expresses a proposition which is perspicuously represented by the corresponding sentence of the form "There is a property P, α possess P and β possesses P". And while Moore does not, here in 1911, embrace this analysis in all cases, he does accept it for the case of number:

> [T]here is a . . . type of cases [sic], in which it seems to me plainer that a universal of my third kind *is* involved, in which it seems to me that we can perhaps distinguish this universal—hold it before our minds, and be sure that it is there. . . . Consider, for instance, the group formed of all collections which are collections of two things and no more—which are pairs or couples. Every pair or couple of things, no matter what the things may be, obviously has

> some property which belongs to all other pairs or couples and to nothing else—*the* property which we express by saying that each of them is a pair or couple. . . . The property in question does seem to consist in the fact that *the number two* belongs to every such collection and only to such a collection; and the number two itself does seem to be a universal of my third kind: something which is neither a relation nor a property which consists in the having of a relation to something or other. And it seems to me that in this case we can perhaps distinguish the universal in question: that we can hold the number two before our minds, and see what it is, and *that* it is, in almost the same way as we can do this with any particular sense–datum that we are directly perceiving. (Ibid., 366)

And he adds shortly thereafter that "the resemblance between the pairs does merely consist in the possession of a common property", that "it is obvious that exactly the same argument applies to each of the other whole numbers", and that "each particular whole number, therefore, does seem to be a universal of my third kind" (ibid., 368). Thus, in these remarks, Moore embraces the "absolute" theory of number, according to which the proposition expressed by an instance of (Num_1) is represented perspicuously by the corresponding instance of (Num_2). Moreover, he indicates here, as does the Moorean Russell, and in accord with (PoA), that considerations as to what is "before our minds" when we understand the relevant sentences suffice for establishing that theory of number.

Consider finally here the following passage from Husserl's *Logical Investigations*:

> [W]e find . . . that whenever things are 'alike', an identity in the strict and true sense is also present. We cannot predicate exact likeness of things, without stating the respect in which they are thus alike. Each exact likeness relates to a Species, under which the objects compared are subsumed. . . . It would of course appear as a total inversion of the true state of things, were one to try to define identity, even in the sensory realm, as being essentially a limiting case of 'alikeness'. Identity is wholly indefinable, whereas 'alikeness' is definable: 'alikeness' is the relation of objects falling under one and the same Species. If one is not allowed to speak of the identity of the Species, of the respect in which there is 'alikeness', talk of 'alikeness' loses its whole basis. (1900–1, Investigation II, Chapter 1, §3, 242)

Thus, like the early Russell and Moore, Husserl holds that "exact likeness" in a certain respect is "definable" as "the relation of objects falling under one and the same Species", so that an instance of (Sim_1), stating of two things that they are exactly alike in a certain respect, expresses a proposition whose full analysis is given by the corresponding instance of (Sim_2), stating of those things that they "fall under one and the same Species". Further, for Husserl, as for the early Russell and Moore, accepting this view follows from applying some such principle as (PoA); for like the early Russell and Moore, Husserl indicates that anyone who understands an instance of (Sim_1) will take it as obvious that what is thus meant is fully expressed in the corresponding instance of (Sim_2).

2. RUSSELL POST–PEANO I: THE TRANSFINITE, EPISTEMOLOGY, AND THE RELATION OF PHILOSOPHY TO SCIENCE

Besides characterizing his attending the Paris Congress of August 1900 as the "the most important event" in "the most important year in my intellectual life", Russell writes (in his *Autobiography*) that "intellectually, the month of September 1900 was the highest point in my life" (1967, 145). Describing that month, he writes:

> The time was one of intellectual intoxication. My sensations resembled those one has after climbing a mountain in a mist, when, on reaching the summit, the mist suddenly clears, and the country becomes visible for forty miles in every direction. For years I had been endeavouring to analyse the fundamental notions of mathematics, such as order and cardinal numbers. Suddenly, in the space of a few weeks, I discovered what appeared to me to be definitive answers to the problems which had baffled me for years. And in the course of discovering these answers, I was introducing a new mathematical technique, by which regions formerly

abandoned to the vaguenesses of philosophers were conquered for the precision of exact formulae. (Ibid.)

In October 1900, Russell drafted "The Logic of Relations", which was published in Peano's journal *Rivista di Matematica* in 1901; in November through December 1900 he wrote final drafts of Parts III–VI of PoM (Parts III–V in November, Part VI in December); and in January 1901, he wrote the popular essay "Recent Work in the Philosophy of Mathematics", which he later reprinted under the title "Mathematics and the Metaphysicians".

There are numerous issues regarding exactly how Russell's views changed in the wake of his attending the Paris Congress in August 1900 and exactly when, following the Congress, Russell came to accept various views that he incorporated in PoM.[15] Here, I make three general points. First, it is clear that during the period of "intellectual intoxication" immediately following the Paris Congress, Russell came to hold—what, as I discuss below, he had formerly denied—that mathematicians such as Dedekind, Weierstrass, and, especially Cantor, had solved all the traditional problems of infinity and continuity. Russell incorporates Cantorian views of transfinite cardinals and ordinals as well as Cantor's account of continuity in "The Logic of Relations", and the work of Dedekind, Weierstrass, and Cantor is central to the argument of Part V of PoM as well as "Recent Work in the Philosophy of Mathematics".

Second, it was not during this initial period following the Paris Congress that Russell came to accept the so–called "Frege–Russell" definitions of cardinal numbers, according to which the cardinal number of a given class α is the class of classes equinumerous with α, and the cardinal number n is the class of n–membered classes. While this view of cardinal numbers appears in the published version of "The Logic of Relations", it does not appear in the October 1900 draft of the paper, nor does it appear in the draft of PoM that Russell composed in November–December 1900; instead, Russell appears to have introduced these definitions sometime between February and June 1901, when he made the final corrections to "The Logic of Relations".[16] Moreover, as I discuss in §3.1 below, as late as May 1902, Russell indicates that while such definitions of cardinal numbers are "formally" acceptable, they are not "philosophically" adequate; and it is only in the final copyediting of PoM, sometime after June 1902 that Russell embraces these definitions, "philosophically" as well as "formally". Regarding these definitions as a philosophically adequate account of the cardinal numbers is a major event in Russell's development; for, in doing so, he is not only rejecting his Moorean absolute theory of number, but also, more generally, has the means to reject all his Moorean absolute theories of order, including his earlier theories of time and magnitude.

Third, it was not until May 1901 that Russell first discovered a version of the paradox that bears his name, and there is a question as to when Russell came to regard it as presenting a deep and fundamental problem. Russell himself writes in his *Autobiography* that "at first I supposed that I should be able to overcome the contradiction quite easily, and that probably there was some trivial error in the reasoning" and that "throughout the latter half of 1901 I supposed the solution would be easy, but by the end of that time I had concluded it was a big job" (1967, 147). Some have argued that it was not until he presented the "contradiction" to Frege in June 1902 and came to see how devastating it was to Frege that Russell came to recognize the full importance of his "contradiction".[17]

In what follows, my primary concern is with the first two of these three post–Peano developments.[18] In particular, in this Part, after discussing (in §2.1) some aspects of how Russell's views of Cantor, Dedekind, and Weierstrass change immediately after the Paris Congress, I argue that his post–Peano acceptance of their work conflicts with a number of features of his overall Moorean philosophy—including its epistemological foundationalism (§2.2), its characterization of "the business of philosophy" (§2.3), and its view of the relation between philosophy and science (§2.4)—and thereby provides the model for his later "scientific conception of philosophy" (§2.5). In Part 3, I argue that Russell's acceptance of the so–called "Frege–Russell" definitions of cardinal numbers is not only incompatible with his earlier absolute theory of number, but is also, opposed to his Moorean conception of analysis and relies on a notion of "vagueness" that threatens to undermine (Aug) and (PoA), so that he cannot present a plausible account of analysis that enables him to regard these definitions of numbers (along with definitions of other mathematical terms he accepts after the Paris Congress) as legitimate until he rejects those Moorean views of meaning and understanding and accepts instead his later behaviorist account of understanding.

2.1. Russell on Cantor, Dedekind and Weierstrass Pre– and Post–Peano

Prior to the Paris Congress, Russell was aware of the work of Cantor, Dedekind, and Weierstrass, but did not regard it as philosophically acceptable. First of all, during his Moorean period, Russell rejects Cantor's theory of transfinite numbers. Thus in his pre–Peano draft of PoM, Russell begins his chapter entitled "Transfinite Numbers" by writing:

> The mathematical theory of infinity may almost be said to begin with Cantor. The infinitesimal calculus, though it employs infinity, contrives to smuggle it out of the results, and deals with it as briefly as possible. ... Cantor has abandoned this cowardly policy, and has brought the skeleton out of its cupboard. He has been emboldened in this course by denying that it is a skeleton: In this however, we shall find reason to disagree. ... I cannot persuade myself that his theory solves any of the philosophical difficulties of infinity, or renders the antinomy of infinite number one whit less formidable. (1899–1900, 119)

In particular, Russell holds that while there are infinite classes and infinite series, unavoidable antinomies result if one holds that there are transfinite numbers—cardinal or ordinal—that may be assigned to such classes and series.

Russell's central argument against recognizing transfinite numbers, to which he alludes in the above passage, is "the antinomy of infinite number", which he presents in an 1899 manuscript as follows:

> *Number in connection with whole and part, quantity, and order*. The application of these ideas leads to (a) all numbers (b) the greatest number (c) the last number. Observe that (b) is improper: it means the number applying to the greatest collection. All three are commonly called infinite number, and imply an antinomy, since their being can be both proved and disproved. (a) the most fundamental: There are many numbers, therefore there is a number of numbers. If this be N, $N + 1$ is also a number, therefore there is no number of numbers. (CP2, 265; see also 1899–1900, 123–5; 1900b, 231)

This argument[19] has the form of the paradox of the largest cardinal: given that every class has a cardinal number of elements, it would seem that there should be a greatest cardinal number, "the number applying to the greatest collection" (which Russell here seems to assume would be the same as "the number of numbers"); however, given that for every number, there is a greater number (a view which Russell here generates by the assumption that given any number N, N+1 will be greater than N[20]), there can be no greatest number. And to avoid this contradiction, Russell rejects the assumption that "a given collection of many terms must contain some definite number of terms", apparently taking that assumption as less obvious than that for any number N, there is a number $N + 1$ greater than N. In particular, since Russell agrees that "*all finite numbers* is a legitimate concept" and that there is no finite number of all finite numbers, he holds that while there are infinite classes, such as the class of all finite numbers, such classes have no number of terms (see 1899–1900, 124–5).

In denying transfinite numbers while countenancing infinite classes, Russell is developing a view he finds in Leibniz. Thus, in PoL, Russell writes that Leibniz accepted "the principle ... that infinite aggregates have no number" and adds that this "principle is perhaps one of the best ways of escaping from the antinomy of infinite number" (117, fn; see also 1900b, 231 and 1946, 784). Further, in writing that "Leibniz denied infinite *number*, and supported his denial by very solid arguments" (PoL, 109), Russell cites the following passage from Leibniz:

> [T]he number of all numbers implies a contradiction, which I show thus: To any number there is a corresponding number equal to its double. Therefore the number of all numbers is not greater than the number of even numbers, *i.e.* the whole is not greater than its part. (Ibid., 244)

Here, Leibniz is, in effect, pointing out that a contradiction arises if one assumes both that sets whose members can be placed in a one–one correspondence with each other (such as the set of natural numbers and the set of even numbers) have the same cardinal number and that a proper subset of a given set must have fewer members than that given set, here alluding to the Euclidean axiom that the part is less than the whole. Leibniz avoids this contradiction by holding that while these principles apply to aggregates that may be assigned a cardinal number, infinite aggregates have no cardinal number; and, prior to the Paris Congress, Russell accepts Leibniz's argument here and his conclusion.

Further, in his pre–Peano draft of PoM, Russell also rejects Dedekind's and Cantor's accounts of irrational numbers as well as Cantor's account of continuity. According to Russell, Dedekind assumes an "axiom" from which it follows that for every convergent sequence of rational numbers,[21] there is a number that is the limit of that sequence; however, for Russell, there is no such axiom that is "possessed of self–evidence" (1899–1900, 115), in which case Dedekind has provided no good reason for countenancing irrational numbers as the limits of certain converging sequences of rational numbers.[22] According to Russell, Cantor bases his account of irrational numbers as well as his account of continuity on his account of transfinite ordinals. Since, for Cantor, for any converging infinite sequence of rational numbers, there will always be the ωth term of that sequence (where ω is Cantor's first transfinite ordinal), which will then be the limit of that sequence. And since the structure of the series of real numbers is the basis for Cantor's account of continuity, Russell holds that without his theory of transfinite ordinals, Cantor has provided no good reason either for holding that there are irrational numbers or for characterizing continuity in terms of the structure of the real numbers. As Russell writes:

> The admission of irrationals, and of continuity in Cantor's sense, would seem to depend wholly upon the admission of the completed infinite, *i.e.* of Cantor's transfinite numbers. If there be a limit [to the series whose nth term is A_n], it is A_ω; if A_ω is not admissible, there is no limit.... If this conclusion be valid, it is not irrationals, but transfinite integers, that introduce a new idea; and this idea is properly that of the completed or definite infinite. (1899–1900, 115)

Thus, since he rejects Cantor's theory of transfinite ordinals, the pre–Peano Russell also rejects Cantor's theory of the irrationals and of continuity.

While Russell thus rejects "the arithmetical theory of irrationals"—the view that irrationals are genuine numbers on a par with the whole numbers and rationals—he goes on to develop a "quantitative" account of the irrationals (ibid., 138) that depends on the view that some quantities are infinitely divisible. In rejecting transfinite numbers, Russell

holds of such quantities that they do not have a "numerically measureable" number of parts, so that the "magnitude of divisibility" associated with such a quantity is "incommensurable" with the "magnitude of divisibility" associated with a quantity that is finitely divisible; however, by recognizing "incommensurable" magnitudes of divisibility, Russell holds that irrational numbers may then be "interpreted" (ibid.) as ratios of such incommensurable magnitudes, ratios which "can only be expressed as limits" (ibid., 137), an approach which, as Russell indicates (ibid., 138), goes back to Definition V of Book V of Euclid's *Elements*.

Russell was aware pre–Peano of the view of "Weierstrass and his followers" that "all pure mathematics should be regarded as dealing exclusively with numbers" so that "the reference to quantity [in pure mathematics] has, in the past thirty years, been wholly eliminated" (ibid., 54); hence, he acknowledges in his pre–Peano draft of PoM that his "quantitative" account of irrationals "places an awkward obstacle in the way of the complete arithmetization of mathematics" (ibid., 138) and "may appear unduly conservative, and may seem to do scant justice to the modern theory of number" (ibid., 140). However, as he continues:

> It [this quantitative account of irrationals] seems, however, to be forced upon us by the difficulties we found in Chapter V [Transfinite Numbers]. These difficulties are not new, but are merely old puzzles worded to suit transfinite numbers. I am unaware of any answer to them, and until such an answer is found, the rejection of infinite number seems unavoidable. Since, nevertheless, infinity is in some sense forced upon us, it is preferable to give independence to the quantitative infinite, and allow this to apply to wholes which are not amenable to number. (Ibid., 140)

For the pre–Peano Russell, it is by "giv[ing] infinity a quantitative meaning" (ibid., 137)—in particular by allowing that there are infinitely divisible quantities but no infinite numbers—that he can provide an "interpretation" of the irrationals without actually countenancing irrational numbers.

By Russell's post–Peano draft of PoM, all this has changed. First, he now embraces Cantorian set theory. Thus, while he begins his chapter entitled "Transfinite Cardinals", as he had begun his earlier draft of "Transfinite Numbers"—by writing that "Cantor has abandoned" the "cowardly policy" of "contriv[ing] to hide [infinity] away" and "has brought the skeleton of the cupboard"—he continues by writing:

> Indeed, like many other skeletons, it was wholly dependent on its cupboard, and vanished in the light of day. Speaking without metaphor, Cantor has established a new branch of Mathematics, in which, by mere correctness of deduction, it is shown that the supposed contradictions of infinity all depend upon extending, to the infinite, results which, while they can be proved concerning finite numbers, are in no sense necessarily true of *all* numbers. (PoM, 304[23])

And since Russell had not yet discovered his paradox, he is fully confident in his post–Peano 1900 draft of PoM that Cantor's theory provides the means to avoid all "the supposed contradictions of infinity".

Further, while he still rejects the way in which Dedekind and Cantor introduce the irrational numbers (and also now criticizes Weierstrass in this regard)—arguing, as he had earlier, that on their theories it "is evidently a sheer assumption" (PoM, 281) that for any convergent series of rational numbers there is a number which is the limit of that series and that any proposed "axiom of continuity" that would guarantee the existence of such a limit has "no vestige of self–evidence" (ibid., 280)—he now (ibid., 271) defines real numbers, rational and irrational, as classes of rational numbers (that he calls "segments") that are neither null nor co–extensive with the rational numbers but which have no greatest member—that is, which are such that for any x which is in segment S, there is a y in S such that $x<y$—and which are such that if y is in S then so is every $x<y$ (ibid., 270).[24] He then argues that "the usual properties of real numbers"—such as that there are more reals than rationals—"belong to segments of rationals" (ibid., 274). Given this definition (to which he was led by reading a paper from Peano—see PoM, 274–5), Russell no longer needs to appeal to his apparently retrograde "quantitative" account of the irrationals; on the contrary, he now regards himself as fulfilling the project of arithmetizing mathematics more fully than do Dedekind, Cantor, or Weierstrass themselves. Thus in defending his definition of real numbers as "segments of rationals", Russell writes:

> [T]here is no logical ground for distinguishing segments of rational numbers from real numbers. If they are to be distinguished, it must be in virtue of some immediate intuition, or of some wholly new axiom, such as, that all series of rationals must have a limit. But this would be fatal to the uniform development of Arithmetic and Analysis from the five premisses which Peano has found sufficient, and would be wholly contrary to the spirit of those who have invented the arithmetical theory of irrationals. My theory [In PoM, "The above theory"}, on the contrary, requires no new axiom, for if there are rationals, there must be segments of rationals; and it removes what seems, mathematically, a wholly unnecessary complication, since, if segments will do all that is required of irrationals, it seems superfluous to introduce a new parallel series with precisely the same mathematical properties. I conclude, then, that an irrational actually *is* a segment of rationals which does not have a limit. . . . (PoM, 286, as collated with Byrd, 1994, 76)

For Russell, that is, while Dedekind, Cantor, and Weierstrass move from rationals to irrationals only by assuming some such "axiom" that every convergent series of rationals has a limit, he has arrived at the irrationals from the rationals without such an axiom, instead assuming only (what those others also accept) that where there are rationals, there are also classes (and hence segments) of rationals. And in that case, Russell's account, is truer "to the spirit of those who have invented the arithmetical theory of irrationals" than are the accounts of Dedekind, Cantor, or Weierstrass themselves.

Moreover, since he now accepts a purely "arithmetical" theory of the real numbers, he can accept an account of continuity that appeals to no "quantitative" considerations. In particular, he now holds that Cantor produces a purely "ordinal" definition of continuity, and, since he now regards the real numbers, construed as segments of rationals, as fully "arithmetical", he claims also that "instances fulfilling the definition [of continuity] may be found in Arithmetic" (PoM, 303). Whereas he previously held that Cantor could not define continuity in his sense without illegitimately assuming the existence of limits of certain series and denied that any "arithmetical" structure is continuous in Cantor's sense, he now regards Cantor's definition as legitimate and the real numbers as an arithmetical series that meets Cantor's definition.

Accordingly, whereas Russell concluded Part V ("Continuity and Infinity") of his pre–Peano draft of PoM by writing:

> [L]et us sum up the discussions of Part V ["Infinity and Continuity"]. ... We ... examined the arithmetical theory of irrationals: we saw that these must be already known to exist, if they are to be determined by series of rational numbers. Their existence, we found could only be proved be means of some axiom of continuity, or by transfinite numbers. Dedekind's suggested axiom was found to be unsatisfactory, and we were left to examine transfinite numbers. As a preliminary, Cantor's definition of continuity was discussed.... It was found that continuity, in Cantor's sense, depends also upon transfinite numbers. ... But when we came to these, we found that all the old difficulties as to the number of numbers remained. We found that Cantor's infinite is as ambiguous and as little definite as the older kinds, and cannot disprove the proposition that every number is finite, (1899–1900, 140)

he concludes his post–Peano November 1900 draft of the same Part by writing:

> To sum up the discussions of this Part: We saw, to begin with, that irrationals are to be defined as those segments of rationals which have no limit, and that in this way analysis is able to dispense with any special axiom of continuity. We saw that it is possible to define the kind of continuity, which belongs to real numbers, in a purely ordinal manner [in PoM, "to define, in a purely ordinal manner, the kind of continuity which belongs to real numbers"], and that continuity so defined is not self–contradictory. Finally we addressed the philosophical questions concerning continuity and infinity.... [W]e found that all the usual arguments, both as to infinity and as to continuity, are fallacious, and that no definite contradiction can be proved concerning either. (PoM, 368, as collated with Byrd, 1994, 86)[25]

It would be hard to overstate the importance for Russell of his coming to embrace the work of Cantor, Dedekind, and Weierstrass; and in his (influential[26]) essay "Recent Work on the Principles of Mathematics", which incorporates many of the central points of Part V of PoM, Russell dramatically characterizes their achievements. Thus, for example, he writes:

> Zeno was concerned ... with three problems.... These are the problems of the infinitesimal, the infinite, and continuity. To state clearly the difficulties involved, was to accomplish perhaps the hardest part of the philosopher's task. This was done by Zeno. From him to our own day, the finest intellects of each generation in turn attacked the problems, but achieved, broadly speaking, nothing. In our own time, however, three men—Weierstrass, Dedekind, and Cantor—have not merely advanced these three problems, but have completely solved them. The solutions, for those acquainted with mathematics, are so clear as to leave no longer the slightest doubt or difficulty. This achievement is probably the greatest of which our age has to boast; and I know of no age (except perhaps the golden age of Greece) which has a more convincing proof to offer of the transcendent genius of its great men. Of the three problems, that of the infinitesimal was solved by Weierstrass; the solution of the other two was begun by Dedekind, and definitively accomplished by Cantor. (1901d, 370)

In the remainder of this Part, I focus on how accepting the work of Cantor, Dedekind, and Weierstrass undermines central aspects of Russell's Moorean philosophy, including his foundationalist epistemology, his characterization of the method of philosophy, and his view of the relation of philosophy to science.

2.2. The Infinite and the Self–Evident

Given that he held prior to the Paris Congress that countenancing transfinite numbers leads to "hopeless contradictions", it is not surprising that in order for Russell to accept Cantor's theory of the transfinite he had to reject some principles he previously regarded as "self–evident".

And in explaining how one might come to deny an apparently "self–evident" principle, Russell presents views opposed to his earlier foundationalist epistemology.

Thus, in "Recent Work in the Principles of Mathematics", in discussing "the importance of symbolism", Russell emphasizes its role in enabling us prove "self–evident" propositions:

> The fact is that symbolism is useful because it makes things difficult. ... What we wish to know is what can be deduced from what. Now, in the beginnings, everything is self–evident; and it is very hard to see whether one self–evident proposition follows from another or not. Obviousness is always the enemy of correctness. Hence we invent some new and difficult symbolism, in which nothing seems obvious. Then we set up certain rules for operating on the symbols, and the whole thing becomes mechanical. In this way we find out what must be taken as premiss and what can be demonstrated or defined. For instance, the whole of Arithmetic and Algebra has been shown to require three indefinable notions and five indemonstrable propositions. But without a symbolism it would have been very hard to find this out. It is so obvious that two and two are four, that we can hardly make ourselves sufficiently skeptical to doubt whether it can be proved. And the same holds in other cases where self–evident things are to be proved. (1901d, 367–8)

Then, in explaining why one would seek to prove self–evident propositions, Russell adds:

> [T]he proof of self–evident propositions may seem, to the uninitiated, a somewhat frivolous occupation. ... But ... since people have tried to prove obvious propositions, they have found that many of them are false. Self–evidence is often a mere will-o'-the-wisp, which is sure to lead us astray if we take it as our guide. For instance, nothing is plainer than that a whole always has more terms than a part, or that a number is increased by adding one to it. But these propositions are now known to be usually false. Most numbers are infinite, and if a number is infinite you may add ones to it as long as you like without disturbing it in the least. (Ibid., 368)

Here, Russell alludes to two "self–evident" propositions that were central to arguments he previously endorsed against admitting transfinite numbers: the proposition that "a whole always has more terms than a part", used in the Leibnizian argument that a contradiction results from attempting to assign transfinite numbers to both the whole numbers and the even numbers; and the proposition that "a number is increased by adding one to it" which was central to Russell's "antinomy of infinite number". Thus, in claiming here that both "these propositions are now known to be usually false" (since the first does not apply to infinite classes and the second does not apply to transfinite cardinal numbers), Russell is admitting, in effect, that in accepting Cantorian set theory, he is rejecting propositions that he had previously taken to be "self–evident". And hence in claiming that "self–evidence is often a mere will-o'-the-wisp, which is sure to lead us astray if we take it as our guide", Russell is also admitting, in effect, that he himself is one who had been thus led astray.

Moreover, in these passages, Russell not only criticizes relying on "self–evidence"; he also presents a method for assessing propositions we take to be "self–evident"—namely, that we should seek to prove them. In particular, he claims that we should avail of a symbolism which is useful just to the extent that it "makes things difficult", thereby making things seem less "obvious", which, in turn, enables us to seek proofs of what we take to be "self–evident". All this is opposed to his Moorean epistemology, according to which it is incoherent to attempt to prove "self–evident" propositions since they just are propositions that admit of no proof from any other proposition. Further, on his Moorean view, there is no sense that "self–evidence" may lead us astray; rather, it is that philosophical theory has led us astray, while "self–evidence"—or, more generally, "immediate perception" or "intuition"—enables us to break the hold of such theory and re–establish contact with reality as it is "in itself". In contrast, Russell's post–Peano project of attempting to prove what he had previously taken to be "indemonstrable" "self–evident" truths by means of a symbolism "in which nothing seems obvious", is designed to avoid the pernicious effects, not of theory, but of "self–evidence".[27]

In his post–Peano draft of PoM, Russell makes more clear how this can be so. In particular, in discussing the "axiom", central to Leibniz's argument against transfinite numbers that "the whole cannot be similar to [that is, have the same number of parts as] the part"(PoM, 359–60), Russell writes:

> [I]t is an axiom doubtless very agreeable to common–sense. But there is no evidence for the axiom except supposed self–evidence, and its admission leads to perfectly precise contradictions [such as the one Leibniz presents]. The axiom is not only useless, but positively destructive, in mathematics, and against its rejection there is nothing to be set except prejudice. It his one of the chief merits of proofs that they instill a certain scepticism as to the result proved. As soon as it was found that the similarity of whole and part could be *proved* to be impossible for every *finite* whole, it became not unplausible to suppose that for infinite wholes, where the impossibility could not be proved, there was in fact no such impossibility. (Ibid., 366)

For Russell, once we cease to take it as an unprovable truth that "the part is less than the whole" and actually prove it in the case of finite "wholes", the question arises as to whether it also holds in the case of "infinite wholes"; and once we recognize that we cannot prove it in that case also, we will be less inclined to regard it as a "self–evident" truth that applies in all cases, and will instead be in a position to question whether it holds in the case of "infinite wholes".[28] That the "axiom" remains unprovable in the case of "infinite wholes" is no longer an occasion for attempting simply to "intuit" whether or not it is true in that case as well, but rather enables us to set aside the apparently "self–evident" as mere "prejudice" and reap the benefits of Cantor's theory. In particular, we will be able to hold that two classes have the same number of elements if and only if their members can be placed in a one–to–one correspondence with each other; to define an infinite class as those whose members can be put in a one–to–one correspondence with the members of one of its proper subsets;[29] and to then avoid Leibniz's argument against infinite numbers by denying that the "axiom" that "the part is less than the whole" applies for infinite classes. As Russell writes: "This property [a 'collection' has when 'it contains as parts other collections which have just as many terms as it has'], which was formerly thought to be a contradiction, is now transformed into a harmless definition of infinity." (1901d, 373)

More generally, in acknowledging that accepting Cantorian set theory requires rejecting some propositions that we take to be "self–evident", Russell admits that "on the subject of infinity it is impossible to avoid conclusions which at first sight appear paradoxical" (ibid., 376). Here, as elsewhere,[30] Russell regards what is "paradoxical" as what is, broadly speaking, counter–intuitive, not, more narrowly, as a contradiction that may be derived from seemingly obvious premises (so that in PoM, he refer to what has come to be called "Russell's paradox" as "the contradiction", not a "paradox"). His argument in favor of Cantorian set theory is not—as his earlier foundationalist epistemology would seem to require—that however "paradoxical" its conclusions may appear, they are deducible from "self–evident" axioms by means of "self–evident" inferences. Rather, it is that however "paradoxical" its conclusions may appear, there are also "paradoxes" involved in rejecting Cantor's theory, and that the overall benefits of accepting the theory make it worthwhile to accept its "paradoxical" aspects.

In particular, he argues that if we accept the "axiom", opposed to Cantorian set theory, that "the part is less than the whole", we are left with no way to respond to Zeno's argument that Achilles can never catch the tortoise. For, since Achilles and the tortoise are both moving, they each occupy different places at different times, in which case, they occupy the same number of places over any given time–period; but if Achilles is to overtake the tortoise, then the tortoise's path will be only part of Achilles', in which case Achilles will be able to catch the tortoise only if the part (the tortoise's path) has just as many points as the whole (Achilles' path). Hence, for Russell, we are faced with a choice between accepting the "self–evident" proposition that "the part is less than the whole" and having no reply to Zeno's paradox, or having a reply to Zeno while rejecting that "self–evident" proposition, thus embracing what he calls in the following passage "the paradox of Cantor":

> The possibility that the whole and part may have the same number of terms is, it must be confessed, shocking to common–sense. Zeno's Achilles ingeniously shows that the opposite view also has shocking consequences; for if whole and part

> cannot be correlated term for term, it does strictly follow that, if two material points travel along the same path, the one following the other, the one which is behind can never catch up. ... Commonsense, therefore, is in a very sorry plight; it must choose between the paradox of Zeno and the paradox of Cantor [that the whole and part may have the same number of terms]. I do not propose to help it, since I consider that, in the face of proofs, it ought to commit suicide in despair. (PoM, 358)

Russell, that is, does not argue that Cantorian set theory really rests on "self–evident axioms", after all; instead, he wants to convince us both that it is not inconsistent, that the consequences of rejecting it are even more "paradoxical" than are those of accepting it, and that accepting it enables us to solve all the traditional problems of infinity and continuity. His argument is not one based on a foundationalist epistemology but is rather a matter of weighing up the costs and benefits of accepting one or the other of two competing theories. Whereas the Moorean Russell presents himself as accepting "self evident" "common sense" views that only philosophers reject, the post–Peano Russell holds that, at least in the area of the infinite, there is no view agreeable to "common sense", so that whatever view we come to accept will not be decided by what is "self evident" to "common sense" but will rather be mediated by theory.

Of course, once he comes to regard his and other set–theoretic "contradictions" as raising fundamental difficulties and before he arrives at what he takes to be an adequate response to those contradictions, Russell cannot be so confident that Cantorian set theory really is consistent or, hence, that it has succeeded in solving all the traditional problems of infinity. What is more relevant to my concerns here, however, is that by resolving to continue, in spite of the contradictions, to attempt to arrive at a set theory which can deliver the benefits of Cantorian theory without any contradictions, Russell becomes even more pronounced in providing anti–foundationalist, "coherentist" justifications for his eventual position. Thus, in his 1907 paper "The Regressive Method of Discovering the Premises of Mathematics", in discussing how to proceed in the face of "the contradictions", Russell indicates that the goal is to arrive at premises which "get the desired consequences without the admixture of demonstrable falsehood" (1907, 279). For Russell, "even where there is

the highest degree of obviousness, we cannot assume we are infallible" and premises become "more nearly certain" if they are part of "a complicated deductive system, many parts of which are obvious" (ibid.), so that "although intrinsic obviousness is the basis of every science, it is never, in a fairly advanced science, the whole of our reason for believing any one proposition of the science" (ibid.).[31] And this is in sharp contrast to the Moorean epistemology, according to which with regard to an "ultimate premise", no reason can be given for its truth, and we accept it only by considering it in "absolute isolation" from every other proposition and "immediately perceiving" its truth.

While some have noted that Russell's post–paradox justification of logical axioms is incompatible with the sort of epistemological foundationalism often associated with him,[32] my point here is that even before Russell was aware of his paradox, his initial post–Peano defense of Cantor's theory was already incompatible with his foundationalist Moorean epistemology. And this tension between his general philosophical commitment to epistemological foundationalism, which he holds is needed to avoid the sort of holistic "coherence" views he associates with absolute idealism, and the anti–foundationalist epistemology he actually uses to defend mathematical and logical theories that he regards as being of the first importance is reflected throughout his later writings.

Thus, for example, in PoP, he seems, on the one hand, to present a classical foundationalist epistemology that is in accord with his Moorean position when he distinguishes "immediate knowledge of truths"—which "may be called *intuitive* knowledge" and where "the truth so known may be called *self–evident* truths"—from "*derivative* knowledge of truths [which] consists of everything that we can deduce from self–evident truths by the use of self–evident principles of deduction" (PoP, 109). On the other hand, he acknowledges that in some cases—as in deducing "two and two are four" from "the general principles of logic"—"the propositions deduced are often just as self–evident as those that were assumed without proof" (ibid., 112), and he adds further that "self–evidence has degrees" (ibid., 117), so that "all our knowledge of truths is infected with *some* degree of doubt, and a theory which ignored this fact would be plainly wrong" (ibid., 135). Further, he allows himself to appeal to coherentist considerations with regard to "instinctive", if not "intuitive", beliefs when he writes:

> All knowledge, we find, must be built upon our instinctive beliefs, and if these are rejected, nothing is left. . . . Philosophy . . . should take care to show that, in the form in which they are finally set forth, our instinctive beliefs do not clash, but form a harmonious system. There can never be any reason for rejecting one instinctive belief except that it clashes with others; thus, if they are found to harmonize, the whole system becomes worthy of acceptance. (ibid., 25)

While the foundationalist rhetoric is a legacy of the general philosophical commitments of his original Moorean break with Idealism, the acknowledgment that no proposition is so "self–evident" that in apprehending it we have an infallible guarantee of truth and the increasing use of coherentist rhetoric begins, I have argued, with his post–Peano acceptance of particular mathematical and logical theories—in the first instance, Cantor's theory of the transfinite.[33]

2.3. *Definition, Proof, and "the Business of Philosophy"*

Not only does Russell's post–Peano practice conflict with his pre–Peano foundationalism; it also conflicts, more generally, with his Moorean view that "the business of philosophy" is to "discover" the "indefinable and the indemonstrable", where "intuition" is the only method available, so that "an idea which can be defined, or a proposition which can be proved, is of only subordinate philosophical interest". In contrast, in his in his post–Peano discussion of the real numbers, continuity, and infinity, Russell emphasizes throughout what is definable and provable, not what is "indefinable" and "indemonstrable".

As I have discussed, both before and after the Paris Congress, Russell rejects Dedekind's and Cantor's accounts of irrational numbers for the reason that he lacks any "intuition" of such numbers and holds that any "axiom" on the basis of which one could deduce the existence of irrational numbers from a sequence of rational numbers lacks "self–evidence". What is new to Russell's position after the Paris Congress is that he now *defines* real numbers as "segments" of rationals, so that he can *prove* (without any "axiom of continuity") that there are real numbers and that the sequence of reals has the structure obeying Cantor's *definition* of continuity; and, for Russell, this result is of fundamental

importance for the philosophy of space and time. For he argues that Cantor's definition of continuity is sufficient for accounting for the continuity of space and time, and thus for justifying the view—consistent with Russell's metaphysical atomism—that space and time have ultimate, simple constituents—namely, points and moments.[34]

Thus, in introducing Cantor's definitions of continuity, Russell writes:

> The notion of continuity has been treated by philosophers, as a rule, as though it were incapable of analysis. They have said many things about it, including the Hegelian dictum that everything discrete is also continuous and *vice versâ*. . . . But as to what they meant by continuity and discreteness, they preserved a discreet and continuous silence; only one thing was evident, that whatever they did mean could not be relevant to mathematics, or to the philosophy of space and time. (PoM, 287)

Then, in the following paragraph, Russell introduces the mathematical treatment of continuity, by writing that before Cantor "it would have been generally thought sufficient" to "call a series continuous if it had a term between any two"[35] and then adding:

> Nevertheless there was reason to surmise, before the time of Cantor, that a higher order of continuity is possible. For, ever since the discovery of incommensurable in Geometry—a discovery of which the proof is set forth [in PoM, "a discovery of which is the proof set forth"] in the tenth Book of Euclid—it was probable that space had continuity of a higher order than that of the rational numbers, which . . . consists in having a term before any two. . . . But that other kind of continuity, which was seen to belong to space, was treated, as Cantor remarks, as a kind of religious dogma, and was exempted from that conceptual analysis which is requisite to its comprehension. Indeed it was often held to show, especially by philosophers, that any subject–matter possessing it was not validly analyzable into elements. Cantor has shown that this view is mistaken, by a precise definition of the kind of continuity which must belong to space. This definition, if it is to be explanatory of space, must, as

> he rightly urges, by effected without any appeal to space. We find, accordingly, in his final definition, only ordinal notions of a general kind, which can be fully exemplified in Arithmetic. (Ibid., 287–8, as collated with Byrd, 1994, 76)

For Russell, then, Hegel and his followers as well as previous mathematicians treated "the kind of continuity which must belong to space" as unanalyzable—that is to say, indefinable—and not amenable to mathematical treatment. And, for Russell, Cantor's achievement lies in his formulating a definition of continuity, which although it is "effected without any appeal to space" characterizes "the kind of continuity which must belong to space", at least insofar as space has the structure which leads to "incommensurables" in geometry.[36]

Further, as Russell suggests in the passage I have just quoted, while those who took continuity to be unanalyzable also typically denied that what is continuous can be composed of elements. As Russell writes later chapter of Part V of PoM:

> It has always been held to be an open question whether the continuum is composed of elements; and even when it has been allowed to contain elements, it has been often held [in PoM, "alleged"] to be not *composed* of these. . . . But all these views are only possible in regard to such continua as those of space and time. The arithmetical continuum is an object created [in PoM, "selected"] by definition, consisting of elements in virtue of that definition, and known to be embodied in at least one instance, namely the segments of rational numbers. I shall maintain in Part VI that spaces afford other instances of the arithmetical continuum. The chief reason for the elaborate and paradoxical theories of space and time and their continuity, which have been constructed by philosophers, has been the supposed contradictions in a continuum composed of elements. The thesis of the present chapter is, that Cantor's continuum is free from contradictions. This thesis, as is evident, must be firmly established, before we can allow the possibility that spatio–temporal continuity may be of Cantor's kind. (Ibid., 347, as collated with Byrd, 1994, 82)

Here again, Russell emphasizes that what is philosophically important in Cantor's work is what he has defined and proved. By defining a purely ordinal notion of continuity that is provably exemplified by the "segments of rational numbers" (that is, the reals as defined by Russell), Cantor shows that what is continuous may be composed of elements (since the segments of rationals are the elements of a continuous series); and insofar as the main motivation for denying that space and time may be composed of elements has been that the very idea of "continuum composed of elements" is contradictory, then Cantor has undermined the main objection to the view that space and time are continua composed of elements.

Just as Russell emphasizes the philosophical importance of Cantor's definition of continuity, so too he emphasizes the importance of Cantor's (and Dedekind's) definitions of infinity. As he writes in "Recent Work on the Principles of Mathematics":

> [T]hough people had talked glibly about infinity ever since the beginnings of Greek thought, nobody had ever thought of asking, What is infinity? If any philosopher had been asked for a definition of infinity, he might have produced some unintelligible rigmarole, but he would certainly not have been able to give a definition that had any meaning at all. Twenty years ago, roughly speaking, Dedekind and Cantor asked this question, and, what is more remarkable, they answered it. The found, that is to say, a perfectly precise definition of an infinite number or an infinite collection of things. This was the first and perhaps the greatest step. (1901d, 372)

Again, in contrast to his pre–Peano characterization of "the business of philosophy", Russell here presents the pre–eminent philosophical achievement as one of definition. Moreover, as I have discussed in the previous section, for Russell, this definition was made possible only by through attempting to prove what was previously regarded as an "indemonstrable" and "self–evident" "axiom"—namely, that "the whole is greater than the part". And as I discuss in Part 3 below, in coming eventually to accept the so–called "Frege–Russell" definitions of the cardinal numbers, Russell goes further in defining what he previously took to be indefinable and proving what he previously took to be axiomatic, thereby presenting the tasks of definition and proof as not merely of "subordinate philosophical interest" but as fundamental to the philosophical enterprise as he now conceives it.

2.4. Philosophy and Science

Given his pre–Peano foundationalism, Russell clearly demarcates the tasks of philosophy and of science. Since the task of the philosopher is to work "backwards" from beliefs already accepted to the "ultimate premises" we have for our current beliefs, while the task of science is to work "forward" to justify new beliefs on the basis of our current beliefs, then nothing the scientist does is relevant to the philosophical task of identifying the "ultimate premises" for any belief, ordinary or scientific.[37]

However, since accepting the work of Cantor, Dedekind, and Weierstrass threatens this foundationalist view of justification, it also undermines Russell's Moorean demarcation between philosophy and science. For Russell, it is precisely because Cantor's mathematical theory upsets what appears "self–evident" that it enables us to rid ourselves of "prejudices" that have stood in the way of solving the traditional problems of infinity and continuity. Hence, after he embraces the work of Cantor, Dedekind, and Weierstrass, Russell holds that philosophers will have to become aware of mathematical theories, if they want to solve these fundamental and longstanding philosophical problems. Thus, for example, in "Recent Work on the Principles of Mathematics", Russell writes:

> It was formerly supposed that infinite numbers, and the mathematical infinite generally, were self–contradictory. But as it was obvious that there were infinities—for example, the number of numbers—the contradictions of infinity seemed unavoidable, and philosophy seemed to have wondered into a "cul–de–sac". This difficulty led to Kant's antinomies, and hence, more or less indirectly, to much of Hegel's dialectic method. Almost all current philosophy is upset by the fact (of which very few philosophers are as yet aware) that all the ancient and respectable contradictions in the notion of the infinite have been once for all disposed of. (1901d, 372)

And similarly in his post–Peano draft of PoM, he writes:

> Of all the philosophers who have inveighed against infinite number, I doubt whether there is one who has known the difference between finite and infinite numbers, (PoM, 192)

whereupon he goes on to present Cantor's account. And, again, in a passage I have quoted in the previous section, Russell writes:

> The chief reason for the elaborate and paradoxical theories of space and time and their continuity, which have been constructed by philosophers, has been the supposed contradictions in a continuum composed of elements. The thesis of the present chapter is, that Cantor's continuum is free from contradictions. (Ibid., 347)

For Russell, that is, Cantor's definitions of infinity and continuity are not merely of mathematical interest; they show that there is nothing inconsistent in those notions and thereby undermine what Russell regards as one the main reasons philosophers have given for supposing that reality is very different from "appearance". For Russell, just as Zeno's paradoxes gave support to Parmenides' monism, so too Kant's antinomies support his view that we can never know reality "as it is in itself", and Hegel's dialectic, based on generating contradictions, supports his "absolute idealism". But for Russell, since all these paths to monism and/or idealism, depend on the view that the notions of infinity and continuity lead to unavoidable contradictions, they are all invalidated by Cantor's theory of the transfinite. And this is a theme to which Russell returns throughout his writings.[38]

Thus, in PoP, Russell writes generally:

> Most of the great ambitious attempts of metaphysicians have proceeded by the attempt to prove that such and such apparent features of the actual world were self–contradictory, and therefore could not be real. (PoP, 145)

As an example, he mentions that while "space and time appear to be infinite in extent and infinitely divisible, . . . philosophers have advanced arguments tending to show that there could be no infinite collections of things", so that "a contradiction emerged between the apparent nature of space and time and the supposed impossibility of infinite collections" (ibid., 146). And after claiming that following Kant "very many philosophers have believed that space and time are mere appearance, not characteristic of the world as it really is", he writes:

> Now, however, owing to the labours of the mathematicians, notably Georg Cantor, it has appeared that the impossibility of infinite collections was a mistake. They are not in fact self–contradictory, but only contradictory of certain rather obstinate mental prejudices. Hence the reasons for regarding space and time as unreal have become inoperative, and one of the great sources of metaphysical constructions is dried up. (Ibid., 147)

In his writings immediately following the Paris Congress, Russell was attempting to convince philosophers of the philosophical importance of the mathematical work of Cantor and others; by the 1920's, he was disdainful of philosophers who had failed to pay attention to such work. Thus, for example, in his 1924 essay "Philosophy in the Twentieth Century", he again presents Zeno, Kant, and Hegel as "manufacturing contradictions which were designed to show that mathematicians had not arrived at real metaphysical truth, and that the philosophers were to supply a better brand" and claims that these sorts of arguments were "destroyed" by the work of nineteenth–century mathematicians, including Cantor, who "invented a theory of continuity and of infinity which did away with all the old paradoxes upon which philosophers had battened". However, for Russell, while "all these results were obtained by ordinary mathematical methods, and were as indubitable as the multiplication table[,] philosophers met the situation by not reading the authors concerned" (1924b, 461–2). Likewise, in 1920, he writes:

> Logic has made, during the last sixty years, greater advances than in the whole previous history of mankind. These advances have all been made by men whose training was predominantly scientific or mathematical, and have been opposed or ignored by orthodox philosophers. . . . [O]fficial academic philosophy, now as at the time of the Renaissance, is engaged in the endeavour to keep alive an antiquated technique, and to ignore the new knowledge which is rendering old problems trivial. Philosophy is associated traditionally with two studies with which it has no essential affinity, namely theology and Greek. If it is to become vital to our universities, it must come to be associated instead with science. But it would be almost as difficult to effect such a change as to carry it through the Social Revolution. (1920b, 405–6)

And in a 1923 review of C. D. Broad's book *Scientific Thought*, Russell writes:

> [Broad] proceeds on the assumption that the business of philosophy is to clear up the fundamental ideas and beliefs of the special sciences. . . . It cannot be denied that there is an important study which has these functions, but whether it should be called "philosophy" may be doubted. Cantor in the last generation showed us what to mean by "infinity" and "continuity"; Einstein in our own time has shown that a physical law must be expressible in tensor form. These were philosophical results according to Mr. Broad's definition, but Cantor and Einstein were not philosophers. The philosophers, in both cases, have done all that lay in their power to prevent the spread of new clear ideas—by fallacious refutation in the first case and fallacious interpretation in the second. On a behaviorist basis, philosophy is to be defined as what a philosopher does. This is not (except in a few cases like Mr. Broad's) what Mr. Broad calls philosophy, which has been left mainly to mathematicians and physicists. I should myself, on behaviorist grounds, define "philosophy" as "the invention of fallacies to conceal or ignorance"; but that would compel me to deny Mr. Broad as a philosopher. (1923b, 260–1)

All these passages reflect the sort of naturalism—opposed to Russell's Moorean conception of philosophy—which is characteristic of Russell's writings in the 1920's, a naturalism according to which the most current theories in mathematics and science may be relevant for addressing philosophical problems.[39] However, in all these passages, Russell alludes to his view, which he first came to accept immediately following the Paris Congress, that the work of mathematicians, most notably Cantor, has direct philosophical bearing. Russell's "naturalism", while pronounced in his post–1918 publications, begins in the final months of 1900.

By 1946, in *A History of Western Philosophy*, in discussing "the philosophy of logical analysis", Russell writes: "The origin of this philosophy is the achievements of mathematicians who set to work to purge their subject of fallacies and slipshod reasoning." (783) And he then goes on to mention Weierstrass, "who showed how to establish the calculus without infinitesimals, and thus at least made it logically secure" (ibid.) and Cantor, "who developed the theory of continuity and infinite number. . . , thereby taking into the realm of exact logic a whole region formerly given over to mysticism and confusion" (ibid., 783–4). Thus, by this point, Russell associates "the philosophy of logical analysis" only with views that he himself came to acquire after the Paris Congress; he does not mention here any views he accepted during his Moorean period that actually led him to break with Idealism.

2.5. The Scientific Method in Philosophy

In addition to coming to hold that the results of mathematical and scientific work should be relevant to philosophers, Russell also comes to emphasize that philosophers should emulate the methods of science and mathematics. Thus, for example, Russell concludes his 1914 Herbert Spencer Lecture, entitled "On Scientific Method in Philosophy", by stating:

> The adoption of scientific method in philosophy, if I am not mistaken, compels us to abandon the hope of solving many of the more ambitious and humanly interesting problems of traditional philosophy. Some of these it relegates, though with little expectation of a successful solution, to special sciences, others it shows to be such as our capacities are essentially incapable of solving. But there remains a large number of the recognized problems of philosophy in regard to which the method advocated gives all those advantages of division into distinct questions, of tentative, partial and progressive advance, and of appeal to principles with which, independently of temperament, all competent students must agree. The failure of philosophy hitherto has been due in the main to haste and ambition: patience and modesty, here as in other sciences, will open the road to solid and durable progress. (1914b, 72–3)

While "the scientific method in philosophy" which Russell presents in this paragraph is meant to be opposed to "heroic" system–building in the tradition of Hegel, it is also fundamentally different from his early Moorean conception of philosophy. Here he presents philosophy as a cooperative enterprise, admitting of "tentative, partial, and progressive advance" and of "appeal to principles with which, independently of temperament, all competent students must agree"; in contrast, for the Moorean Russell, philosophy as concerned with matters that are so fundamental—the "indefinable" and the "indemonstrable"—that with regard to them no rational argument is possible and each individual philosopher can rely only on his or her own "intuition" or "immediate perception". For the Moorean Russell, philosophy is not characterized by "tentative, partial, and progressive advance" but rather by acts of immediate insight into the ultimate constituents of reality. Nor for the Moorean Russell, is there any concern as such with finding "principles with which, independently of temperament, all competent students must agree"; rather, the Moorean philosopher acknowledges that because philosophy is fundamentally a matter of "intuition", not argument, philosophy is "in a certain impersonal sense controversial", since there is no guarantee that "all competent students" will "intuit" the same "indefinables" and "indemonstrables".[40]

In his Herbert Spencer Lecture, Russell does not cite Cantor's approach to the traditional problems of infinity and continuity as exemplifying "the scientific method in philosophy";[41] however, in other places he does. Thus, in the same year as he gave the Herbert Spencer Lecture, he gave lectures at Harvard published under the title *Our Knowledge of the World as a Field for Scientific Method in Philosophy* (OKEW), in which he devotes three of the eight lectures to continuity and infinity. In particular, Russell begins Lecture VII ("The Positive Theory of Infinity") by writing that "the positive theory of infinity" is "among the triumphs of scientific method in philosophy", and so is "especially suitable for illustrating the logical–analytic character of that method" (OKEW, 185). And inin the final parin the in the final paragraph of the book, he writes:

> [T]o the large and still growing body of men engaged in

> the pursuit of science—men who hitherto, not without justification, have turned aside from philosophy with a certain contempt—the new method, successful already in such time–honoured problems as number, infinity, continuity, space and time, should make an appeal which the older methods have wholly failed to make. (Ibid., 242)

For Russell, to adopt the "scientific method in philosophy" is to apply to all areas of philosophy the same style of thinking that he holds enabled Cantor to solve the traditional problems of infinity and continuity.

Similarly, three years earlier, Russell concludes his lecture "Analytic Realism" by claiming:

> We know now that all the past difficulties in the notions of the infinite and the continuum disappear when the methods of Weierstrass and Cantor are used. Curiously enough, however the kinds of paradoxes already known to the Greeks, and named *Insolubilia*, and which were believed to be nothing but trivial amusements, have abruptly surfaced in mathematical logic. ... Since then the enemies of novelty have contented themselves in saying that it would be better to think about something else, whereas mathematicians did not, in general, possess the knowledge of philosophy and logic to solve such problems. However, I believe I have solved this difficulty by recognizing the existence of a hierarchy of different logical types. I will not attempt to explain the theory of types. Suffice it to say that it is contrary to the philosophical spirit as well as the scientific spirit to divert a train of thought in a certain field because the solutions to its problems are not immediately forthcoming. I, for one, do not believe that there are any "antinomies". Contradictions are not mistakes, and to solve them requires only patience and analytic ingenuity. Heroic solutions have been abused in philosophy, detailed work has been neglected, and there has been too little patience. ... The true method, in philosophy as in science, should be inductive, meticulous, respectful of detail, and should reject the belief that it is the duty of each philosopher to solve all problems by himself. It is this method which has inspired analytic realism, and it

> is the only method, if I am not mistaken, with which philosophy will succeed in obtaining results as solid as those obtained in science. (1911a, 138–9)

Here Russell presents the work of Weierstrass and Cantor on infinity and continuity and his own ensuing work on the paradoxes as exemplars of what he would later describe as "the scientific method in philosophy". For Russell, the long process that led from the introduction of the calculus by Newton and Leibniz to the theories of Weierstrass and Cantor that he regards as underpinning the calculus, and, on a smaller scale, the process, the led from his own discovery of the "contradiction" to his favored solution in PM, should be the model for philosophy. In both cases, the method used was neither theorizing in the style of Idealist philosophers that has its conclusion that there are inevitable "antinomies" in these areas of thought nor the Moorean method, which relies on "intuition" of the "indefinable" and "indemonstrable". Rather, what characterizes them are "patience", "detailed work", "analytical ingenuity"—where the focus is on definition and proof—and an "inductive" willingness to accept "axioms", not on the basis of "self–evidence", but rather by weighing up their various costs and benefits.

Above, in the Introduction, I quoted a passage from Russell's 1924 essay "Logical Atomism" in which he writes:

> I began to think it probable that philosophy had erred in adopting heroic remedies for intellectual difficulties, and that solutions were to be found merely by greater care and accuracy. This view I have come to hold more and more strongly as time went on, and it has led me to doubt whether philosophy, as a study distinct from science and possessed of a method of its own, is anything more than an unfortunate legacy from theology. (1924a, 163)

In the first sentence, Russell alludes to his conception of "the scientific method in philosophy"; in the second, he makes the "naturalist" suggestion denying that philosophy is "distinct from science" in either content or method. And in the context in which this passage appears, Russell presents his acceptance of these views of philosophy as a post–Peano development.

For in that essay, after he mentions that he broke from idealism "about 1898 . . . largely as a result of arguments with G. E. Moore" and discusses his (pre–Peano) work on Leibniz, Russell continues that he

> . . . returned to the problem which had originally led me to philosophy, namely the foundations of mathematics, applying to it a new logic derived largely from Peano and Frege, which proved (at least, so I believe) far more fruitful than that of traditional philosophy.
>
> I found that many of the stock philosophical arguments about mathematics (derived in the main from Kant) had been rendered invalid by the progress of mathematics in the meanwhile. Non–Euclidean geometry had undermined the argument of the transcendental aesthetic. Weierstrass had shown that the differential and integral calculus do not require the conception of the infinitesimal, and that, therefore, all that had been said by philosophers on such subjects as the continuity of space and time and motion must be regarded as sheer error. Cantor freed the conception of infinite number from contradiction, and thus disposed of Kant's antinomies as well as many of Hegel's. Finally, Frege showed in detail how arithmetic can be deduced from pure logic. . . . As all these results were obtained, not by any heroic method, but by patient detailed reasoning. I began to think it probable that philosophy had erred in adopting heroic remedies. . . . (Ibid., 162–3)

Thus Russell locates the source not only of his view of "the scientific method in philosophy", which becomes prominent in his writings only by 1914, but also of his more thoroughgoing naturalism, which becomes prominent only in the 1920's, in his post–Peano work in "the foundations of mathematics". And here Russell's retrospective account seems accurate; for, as I have discussed, in the months immediately following the Paris Congress, Russell presents the work of Cantor, Dedekind, and Weierstrass as upsetting his earlier views of the role of "self–evidence" in philosophy, as solving the traditional philosophical problems of the infinity and continuity, and as doing so by employing mathematical, rather than characteristically philosophical, techniques. It is not that

the logical techniques Russell gained in the aftermath of the Paris Congress simply enabled him to articulate and defend "fundamental doctrines" he accepted in his early post–Idealist Moorean period; rather, the post–Peano Russell's engagement with the work of mathematicians undermines his earlier foundationalism and, with it, his earlier views of the method of philosophy and of the relation of philosophy to science.

3. RUSSELL POST–PEANO II: DEFINING THE CARDINAL NUMBERS, ANALYSIS, VAGUENESS, AND BEHAVIORISM

As I discussed in §1.6, during his Moorean period, Russell advocates an "absolute" theory of number, according to which each cardinal number is indefinable; in contrast, by PoM, Russell regards each cardinal number as definable. In particular, he accepts the following definition of the cardinal number of class α:

(Num$_{\text{df}}$) The cardinal number of $\alpha =_{\text{df}} \{\omega\colon \omega \text{ is similar to } \alpha\}$,

where two classes are similar, or equal in number, if and only if the members of the first class can be put into a one–to–one correspondence with the members of the second.[42] As Russell writes, "we decide to identify the number of a class with the whole class of classes similar to the given class" (PoM, 305; see also 115). Further, he defines 0 as the cardinal number of the null–class, that is, the class of classes similar to the null–class (and hence as the class whose only member is the null class); 1 as the class of all classes with a single member; and accepts definitions from which it follows that the cardinal number n is the class of all classes with n members (ibid., 128). More generally, he defines cardinal number, so that "a number is the class of all classes similar to any one of themselves" (ibid.).

In PoM, Russell presents these definitions as central to the logicism he defends in that work—that is, to the view that "all Mathematics is Symbolic Logic" (ibid., 5), a claim that, for Russell, requires that "all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts" (ibid., "Preface", first paragraph). The Moorean Russell is thus not a logicist; for the Moorean Russell regards each cardinal number as an indefinable of arithmetic and hence not as definable in terms of "logical concepts". In contrast, it is incumbent on Russell in PoM to establish that the theory of cardinal numbers involves no non–logical indefinables; and hence if he can show that the notions he uses in the definitions in his theory of the cardinal integers—including, for example, the notions of *class* and of *similarity* that are involved in (Num_{df})—are themselves either indefinables of logic (as Russell in PoM takes *class* to be) or are themselves definable in purely logical terms (as he takes *similarity* to be), then he will be "confirmed . . . in the opinion that Arithmetic contains no indefinables . . . beyond those of general logic" (ibid., 152).

My concern here, however, is not with the relation between his definitions of cardinal numbers and his logicism, or, more generally, with the logicism he defends in PoM;[43] rather, it is with the conception of analysis that would enable Russell to regard those definitions as philosophically adequate. In particular, after discussing (in §3.1) how Russell's PoM account of cardinal number is not only fundamentally opposed to his earlier "absolute" theory of number, but also, more generally, provides him with the means to undermine his other Moorean absolute theories of order, I argue (in §3.2) that Russell's defense of this PoM definitions of cardinal numbers is not only incompatible with his Moorean conception of analysis, but also appeals to a notion of "vagueness" which, if combined with (Aug) and (PoA), leads to the implausible conclusion that, as they are ordinarily used prior to analysis, numerical expressions are meaningless and cannot be understood; and I argue further (in §3.3) it only after he adopts his behaviorist account of meaning and understanding in his post–1918 publications that Russell is in a position to reject (Aug) and (PoA), thereby enabling him accept that notion of "vagueness" along with his post–Peano view of analysis without also accepting that extreme and implausible conclusion.

3.1. From the "Axiom of Abstraction" to the "Principle which Dispenses with Abstraction"

In advocating "absolute" theories of time, magnitude, and number, the Moorean Russell holds that in each case, there is an "independent" or "self–sufficient" series of "intrinsically ordered" absolute positions—moments, magnitudes, or numbers—against which other entities—events, quantities, or classes—are ordered by "correlation". In each case, a sentence of the form

(Ab_1) $E(\alpha,\beta)$,

where "E" is replaced by an expression for a symmetrical transitive relation—"is simultaneous with", "is equal in magnitude with", or "is equal in number with", respectively—expresses a proposition whose perspicuous representation is given by a corresponding sentence of the form

(Ab_2) $(\exists x)(R(\alpha, x)\ \&\ R(\beta, x))$,

where "R" is replaced by an expression for the appropriate many–one relation—namely, "occurs at", or "has", or "possesses". For the Moorean Russell, there are no such indefinable symmetrical transitive relations—no such relations among the ultimate constituents of the universe; instead, to say of two entities that they stand in such a symmetrical transitive relation to each other is to say that they are each borne by the same many–one relation to the same indefinable "absolute position".

In the draft of Parts III–VI of PoM that he composed in the immediate aftermath of the Paris Congress, Russell continued to accept all these views. Thus, in his November, 1900 draft of Part III, as in his pre–Peano draft of PoM, he distinguishes relative and absolute theories of magnitude, and defends the absolute theory, claiming not only that "a direct inspection of what we mean when we say that two terms are equal or unequal" in magnitude will support the absolute theory, and also that while the relative theory of magnitude "is complicated and paradoxical", "both the complication and the paradox . . . are entirely absent in the absolute theory" (PoM, 164), thus suggesting, as he had before, that (PoA) along with "common sense"(which is opposed to "paradox" in the sense Russell here uses that term) supports the absolute theory of magnitude. More generally, he writes:

> The decision between the absolute and relative theories [of magnitude] can be made at once by appealing to a certain general principle, of very wide application, which I propose to call the axiom [in PoM, "principle"] of Abstraction. (Ibid., 166, as collated with Byrd, 1996–7, 160)

As I have mentioned above, what Russell here calls the "axiom of abstraction" is the view that an instance of (Ab_1) expresses a proposition that is perspicuously represented by the corresponding instance of

(Ab$_2$). For Russell, this "axiom" is "merely a careful statement of a very common assumption", namely that symmetric transitive relations "are always constituted by possession of a common property"(ibid.). Further, in Part IV ("Order"), Russell uses the "axiom of abstraction" to argue that there are no series in which—as on the relative theories of magnitude and time—"position is merely relative" (ibid. 220–1, as collated with Byrd, 165–6). For holding that "position is merely relative" requires countenancing primitive symmetric transitive relations—such as *being equal in magnitude* or *simultaneity*—which are ruled out by the "axiom of abstraction". Russell comments that this point is "central to the whole philosophy of space and time"; and he uses it to defend absolute theories of time and space.

Likewise, in his November 1900 draft of Part V of PoM, Russell writes:

> In fact number is a primitive idea, and it is a primitive proposition that every collection has a number. It is therefore philosophically correct that a specification of number should not be by formal definition. (Byrd, 1994, 77–8)

And he adds shortly thereafter:

> [P]hilosophically we must remark that the relation of similarity is complex, and presupposes the cardinal integers, which are therefore not, in the philosophical sense of the word *defined* by means of similarity. The cardinal integers, finite and transfinite alike, are logically independent of classes, which have to them the same kind of relation as quantities have to magnitudes. (Ibid., 78)

Here, Russell indicates that just as he accepts the "absolute" theory of magnitude—according to which magnitudes are indefinables, independent of quantities, which must be invoked to define the relation of *equality in magnitude* between quantities—so too he accepts the "absolute" theory of number— according to which cardinal numbers are indefinables, independent of classes, which must be invoked to define the relation of *similarity* between classes.

Further, even after introducing (Num$_{df}$) in his paper "The Logic of Relations" (sometime in the spring of 1901), Russell continues to hold

that "philosophically", if not "formally", numbers are indefinable. Thus, in his final draft of Part II of PoM written, it seems, in May 1902, Russell writes that "for formal purposes, numbers may be taken to be classes of similar classes", but then provides an argument intended to show that

> Numbers, it would seem, are ... philosophically, not formally indefinable. ... [T]hese indefinable entities are different from the classes of classes which it is convenient to call numbers in mathematics. (Byrd, 1987, 69)

And it is only during his copyediting of page proofs, sometime after June 1902, that Russell changes this passage to read:

> Numbers are classes of classes, namely of all classes similar to a given class. ... [N]o philosophical argument could overthrow the mathematical theory of cardinal numbers set forth [above]. (PoM, 136)

Hence, as late as May 1902, Russell distinguishes each cardinal number from the class of similar classes he identifies it with in PoM as published.

In finally coming to hold that it is philosophically acceptable to regard cardinal numbers as classes of similar classes, Russell has rejected the "absolute" theory of number. No longer are numbers ultimate constituents of the universe constituting a domain separate from classes; now they simply are classes of similar classes. No longer are numbers regarded as indefinable entities in terms of which the relation of *similarity* (or *being equal in number*) between classes is defined; now, that relation of *similarity* is used to define number. No longer is the proposition expressed by an instance of

(Num$_1$) Class α is similar to class β ,

perspicuously represented by the corresponding instance of

(Num$_2$) There is a number n such that α possesses n and β possesses n.

For, given (Num$_{df}$), to say of two classes that they possess the same cardinal number is really to say that the class of classes similar to the first is identical to the class of classes similar to the second, so that an instance of (Num$_2$) is not a perspicuous representation of the proposition it expresses, but rather expresses a proposition that is more[44] perspicuously represented by the corresponding instance of

(Num$_3$) $\{\omega\colon \omega$ is similar to $\alpha\} = \{\omega\colon \omega$ is similar to $\beta\}$.

Further, on Russell's view in PoM, to say of two classes that they are similar is not to invoke the notion of cardinal number, either as indefinables or as classes of similar classes but is rather to say that there is a one–to–one correspondence between the members of those classes; and in that case, an instance of (Num$_1$) expresses a distinct proposition from that expressed by corresponding instances of (Num$_2$) and (Num$_3$). Thus, Russell has gone from holding that corresponding instances of (Num$_1$) and (Num$_2$) express the same proposition—where the latter, but not the former, do so perspicuously—while corresponding instances of (Num$_3$) express distinct propositions (since indefinable numbers are different from classes of similar classes) to holding that corresponding instances of (Num$_2$) and (Num$_3$) express the same proposition—where the latter, but not the former, do so perspicuously—while corresponding instances of (Num$_1$) express distinct propositions.

Moreover, in PoM, Russell recognizes that he can introduce definitions similar to (Num$_{df}$) that will likewise undermine other "absolute" theories of order. In particular, in the course of defending his definitions of cardinal numbers, Russell writes:

> Wherever Mathematics derives a common property from a reflexive, symmetrical, and transitive relation, all mathematical purposes of the supposed common property are completely served when it is replaced by the class of terms having the given relation to a given term; and this is precisely the case presented by cardinal numbers. (Ibid., 116)

Here, Russell indicates that whenever one holds—as his former "axiom of abstraction" requires—that a symmetric transitive relation indicates the possession of a "common property", one may "replace" the "supposed common property" by "the class of terms have the given [symmetric transitive[45]] relation to a given term". As Russell indicates this is exactly the procedure he follows in introducing (Num$_{df}$): instead of regarding the cardinal number of a given class α as an indefinable "property" common to α and any class similar to α, regard it as the class of "terms"—here, classes—having the given relation—here, *similarity*—to the given term—here, class α. In the case of magnitudes, this would amount to accepting

(Mag$_{df}$) The magnitude of quantity $q = \{x\colon x$ is equal in magnitude to $q\}$,

so that the magnitude of a given quantity q is regarded, not as an indefinable "property" common to q and any quantity equal in magnitude to q, but rather as the class of quantities equal in magnitude to q; and Russell introduces this view of magnitude in a footnote he adds to Part III of PoM in the final changes he makes proofreading the typescript (see PoM, 167, as collated with Byrd, 1996–7, 161; see also 1924a, 165). With regard to time, this would amount to accepting

(Time$_{df}$) The moment at which event e occurs $= \{x\colon x$ is simultaneous with $e\}$,

so that the moment at which even e occurs is regarded, not at an indefinable at which e and any event simultaneous with e occurs, but rather as the class of events simultaneous with e, a view of the sort he comes to develop in *Our Knowledge of the External World*.[46]

More generally, Russell is indicating in this passage from PoM that, instead of holding—as his earlier "axiom of abstraction" requires—that an instance of (Ab$_1$) expresses a proposition that is perspicuously represented by the corresponding instance of (Ab$_2$), we can introduce, in place of the "supposed common property" quantified over in the instance of (Ab$_2$), the following definition:

(Ab$_{df}$) $f(\alpha) =_{df} \{x\colon E(x, \alpha)\}$,

where the relevant function f is related to the relevant many–one relation R in the corresponding instance of (Ab$_2$) so that $f(\alpha) = x$ if and only if $R(\alpha, x)$. Given such a definition, an instance of (Ab$_2$) is no longer a perspicuous representation of the proposition it expresses; instead, that proposition is perspicuously represented by the corresponding instance of

(Ab$_3$) $\{x\colon E(x, \alpha)\} = \{x\colon E(x, \beta)\}$,

while the corresponding instance of (Ab$_1$) expresses a distinct proposition.[47]

From a philosophical point of view, the introduction of definitions of the form (Ab$_{df}$) eliminates certain indefinables—namely, indefinable "absolute positions" of the sort that he had previously taken numbers, magnitudes, and moments to be. And it is this point that Russell emphasizes when he discusses what he now calls "the principle of abstraction". As he writes in *Our Knowledge of the External World*:

> The principle, which might equally well be called "the principle which dispenses with abstraction," ... is one which clears away incredible accumulations of metaphysical lumber. ... When a group of objects have that kind of similarity which we are inclined to attribute to possession of a common quality, the principle in question shows that membership of the group will serve all the purposes of the supposed common quality, and that therefore, unless some common quality is actually known, the group or class of similar objects may be used to replace the common quality. (OKEW, 42)

Here, in language similar to that from the passage from PoM I have quoted above, Russell indicates that because adopting definitions of the form ($\mathrm{Ab_{df}}$) enables one to avoid assuming that whenever entities bear to each other a symmetric transitive relation, there is a further indefinable entity—a "common quality"—that is common to the original entities in question, adopting such definitions "clears away incredible accumulations of metaphysical lumber".[48] What he does not mention is that he himself was one who admitted the sort of "metaphysical lumber" that he is now "dispensing with", and that in accepting definitions of the form ($\mathrm{Ab_{df}}$), he is, in effect, rejecting absolute theories of order that had been central to his Moorean philosophy.

From a technical point of view, the introduction of the definitions of the form ($\mathrm{Ab_{df}}$) enables Russell to prove what he had previously taken to be axiomatic—namely, that if an instance ($\mathrm{Ab_1}$) is true, then so is the corresponding instance of ($\mathrm{Ab_2}$). Since he previously held that it is "self–evident" that the corresponding instance of ($\mathrm{Ab_2}$) expresses, perspicuously, the same proposition that the instance of ($\mathrm{Ab_1}$) expresses non–perspicuously, he previously took it to be "self–evident" that if the instance of ($\mathrm{Ab_1}$) is true, then so is the instance of ($\mathrm{Ab_2}$). However, since it is provable (given naïve set theory) that if an instance of ($\mathrm{Ab_1}$) is true, then so is the corresponding instance of ($\mathrm{Ab_3}$),[49] and since, given the definition of the form ($\mathrm{Ab_{df}}$), the instance of ($\mathrm{Ab_3}$) expresses (PErspicuously) the same proposition that is expressed (non–perspicuously) by the corresponding instance of ($\mathrm{Ab_2}$), then, for Russell, it is now provable that if the instance of ($\mathrm{Ab_1}$) is true, then so is the instance of ($\mathrm{Ab_2}$).

The technical advantages of definitions of the form ($\mathrm{Ab_{df}}$) that he introduces in the final stages of completing PoM are thus similar to the technical advantages of the definitions of the irrational numbers that he had introduced earlier, in his November 1900 draft of PoM. By defining irrationals as "segments" of rationals, Russell can avoid regarding irrationals as indefinables whose existence is to be guaranteed by an "axiom of continuity" of the sort he attributes to Dedekind and can also prove—what, for one who accepts such an "axiom of continuity" is unprovable—that for each convergent series[50] of rational numbers there is a real number that is the limit of that series. Similarly, accepting definitions of the form ($\mathrm{Ab_{df}}$) avoids having to regard cardinal numbers, magnitudes, and moments as indefinables whose existence is to be guaranteed by the "axiom of abstraction" and enables him to prove propositions he previously took to be unprovable.

There is, however, a central difference, for Russell, between the definitions of irrational numbers he accepts shortly after the Paris Congress, and the definitions of the form ($\mathrm{Ab_{df}}$) he accepts some time after that. Before accepting those definitions of the irrational numbers, Russell did not hold that there are any such numbers and rejected any "axiom" that would guarantee their existence; in contrast, before accepting definitions of the form ($\mathrm{Ab_{df}}$), he held that there are indefinable "absolute positions" of the sort that those definitions enable us to avoid assuming, and he accepted the "axiom of abstraction" that would guarantee the existence of such "absolute positions". Moreover, for the Moorean Russell, accepting such indefinable "absolute positions" is in accord with the Moorean conception of analysis, for he had claimed that mere "inspection" of "what we mean" when we make claims of the form ($\mathrm{Ab_1}$) supports the view that there are such "absolute positions". As I argue now, the post–Peano Russell does not likewise attempt to reconcile his introduction of definitions of the form ($\mathrm{Ab_{df}}$) with his Moorean conception of analysis; and this raises a problem as to what conception of analysis—and more generally what conception of meaning and understanding—will support his introduction of such definitions.

3.2. *(Num_{df}), Analysis, and Vagueness*

The Moorean Russell took his defense of the absolute theory of number to be in accord with (PoA); for he held that it is obvious that "what we mean" or what is "present to our minds" when we understand, for example, a sentence of the form (Num_1) is a proposition whose perspicuous representation is given by the corresponding sentence of the form (Num_2). However, when he rejects the absolute theory of number and accepts (Num_{df}), he does not likewise present his new account as being in accord with (PoA). That is, it is not that he continues to hold that analyzing propositions expressed by sentences containing numerical expressions is a matter of articulating what is "present to our minds" when we understand such sentences, but changes his view as to what is, in fact, thus "present to our minds"; rather, he has changed his view as to what is required of a philosophically adequate analysis. And in doing so, he presents views of numerical expressions that threaten to undermine not only (PoA) but also (Aug).

Thus, in PoM, after introducing (Num_{df}), Russell writes:

> To regard a number as a class of classes must appear, at first sight, a wholly indefensible paradox. Thus Peano (F, 1901, §32) remarks that "we cannot identify the number of [a class] *A* with the class of classes in question [i.e. the class of classes similar to *A*], for these objects have different properties." He does not tell us what these properties are, and for my part I am unable to discover them.[51] Probably it appeared to him immediately evident that a number is not a class of classes. (PoM, 115)

Here—and in keeping with his early terminology according to which what is "paradoxical" is not necessarily contradictory but is rather, more generally, counter–intuitive—Russell acknowledges that his new account of numbers is far from obvious by direct "inspection"; on the contrary, he concedes that someone might find it "immediately evident that a number is not a class of classes". Earlier, in defending absolute theories of order, he claimed that they obviously reflect 'what we mean" by sentences of the form (Ab_1) and are in accord with "common sense", while it would be "preposterous" and "paradoxical" to accept relative theories of order. Now, he defends a relative theory of number, even

though he readily concedes that it "must appear, at first sight, a wholly indefensible paradox".

Further, Russell adds shortly thereafter:

> Mathematically, a number is nothing but a class of similar classes: this definition allows the deduction of all the usual properties of numbers. . . . But philosophically we may admit that every collection of similar classes has some common predicate applicable to no entities except the classes in question, and if we can find, by inspection, that there is a certain class of such common predicates, of which one and only one applies to each collection of similar classes, then we may, if we see fit, call this particular class of predicates the class of numbers. For my part, I do not know whether there is such a class of predicates, and I do know that, if there be such a class, it is wholly irrelevant to Mathematics.. . . For the future, therefore, I shall adhere to the above definition, since it is at once precise and adequate to all mathematical uses. (Ibid., 116)

Here, Russell is, in effect, comparing his new view of cardinal numbers as classes of similar classes with his earlier view of the cardinals as indefinable "predicates"[52] common to similar classes. He does not claim that his new view reflects more accurately than his old view "what we mean" when we make claims involving cardinal number. He does not even deny that there are indefinable "predicates" of classes of the sort that, on his earlier absolute theory of numbers, *are* the cardinal numbers. Instead, he claims that whether or not there are such indefinable "predicates", regarding cardinal numbers as classes of similar classes "allows the deduction of all the usual properties of numbers", in which case there is no need to address the issue as to whether there are indefinables of the sort that he previously took numbers to be.

Moreover, by writing that his definition of cardinal numbers "is at once precise and adequate to all mathematical uses", Russell is suggesting that our ordinary use of numerical expressions is not likewise "precise". Indeed, in his "Preface" to PoM, Russell appeals to a distinction between "vagueness" and "precision" to justify many of his definitions in that work. In particular, shortly after writing:

> Many words will be found, in the course of discussion, to be defined in senses apparently departing widely from common usage. Such departures, I must ask the reader to believe, are never wanton, but have been made with great reluctance, (PoM, Preface, third–to–last paragraph)

he defends the apparent "departures from common usage" in the case of his definitions of mathematical terms by writing:

> As regards mathematical terms, the necessity for establishing the existence–theorem in each case—*i.e.* the proof that there are entities of the kind in question—has led to many definitions which appear widely different from the notions usually attached to the terms in question. Instances of this are the definitions of cardinal, ordinal and complex numbers. In the two former of these, and in many other cases, the definition as a class, derived from the principle of abstraction, is mainly recommended by the fact that it leaves no doubt as to the existence–theorem. But in many instances of such apparent departure from usage, it may be doubted whether more has been done than to give precision to a notion which had hitherto been more or less vague. (Ibid.)

Here Russell not only emphasizes that definitions "derived from the principle of abstraction" enable one to prove and thereby "leave[] no doubt"—what he would otherwise have take as unprovable and as ascertained only by "intuition"—"that there are entities of the kind in question"; he also indicates that while such definitions "appear widely different from the notions usually attached to the terms in question", they "give precision to . . . notion[s] which had hitherto been more or less vague".

By indicating that our ordinary notion of cardinal number is "more or less vague", Russell is indicating that, as we ordinarily use it, a numerical expression does not yet succeed in standing for any one entity that is correctly regarded as "the meaning" of that expression; and in that case, we have leeway as to which entity to assign to that expression. Thus, for Russell, if there are indefinables of the sort he previously took the cardinal numbers to be as well as classes of similar classes, then

assigning either sort of entity as the reference of numerical expressions will be "adequate to all mathematical uses" of those expressions. And the reason Russell chooses to he chooses to identify cardinal numbers with classes of similar classes rather than indefinable properties common to similar classes is not because he positively denies that there are such indefinables, but rather because he is more sure that there are classes of similar classes. For Russell whether or not there are such indefinables, there are at least classes of similar classes, and that is enough to insure that the mathematical statements we take to be true are, in fact, true. But on this view, analysis is no longer a matter of recognizing "the meaning" an expression has—the entity it stands for—in virtue of its being meaningful at all, "the meaning" with which we must be acquainted in order to understand a sentence containing that expression; rather, it is a matter of assigning, to an expression that was previously vague, a precise meaning that insures that the sentences containing that expression have the truth–values we take them as having.

In thus describing our ordinary use of numerical expressions as "vague", Russell not only presents a view of analysis that is different from his earlier Moorean conception; he is also putting pressure on his official commitment to (Aug) and (PoA). For if a numerical expression, as we ordinarily use it, does not yet succeed in standing for any one entity that is correctly regarded as "the meaning" of that expression then, by the standards of (Aug) and (PoA), that expression, as we ordinarily use it, is not yet meaningful and cannot occur in a sentence that we understand. For if there is no one entity that is "the meaning" of an expression, as it is ordinarily used, then, by (Aug), that expression, as so used, is not yet meaningful. Likewise, if there is no one entity that is "the meaning" of a given numerical expression, as it is ordinarily used, then, there is no entity that is "the meaning" of that expression, as so used, for us to be acquainted with, in which case, by (PoA), no sentence containing that expression, as so used, can be understood. By the standards of (Aug) and (PoA), that is, in "giv[ing] precision to" what "had hitherto been more or less vague", Russell is thereby giving meaning to what was previously meaningless, and making understandable what previously could not be understood; and this would have the extreme and implausible conclusion that in their ordinary, pre–analytic usage, numerical expressions are meaningless and cannot be understood. And while Russell does not draw this conclusion, he consistently accepts presents a view of analysis and employs a notion of vagueness that, when combined with (Aug) and (PoA), leads to that conclusion.

Thus, in *Principia Mathematica*, Volume II, in defending (Num_{df}), Whitehead and Russell write:

> The chief merits of this definition are (1) that the formal properties which we expect numbers to have result from it; (2) that unless we adopt this definition or some more complicated and practically equivalent definition, it is necessary to regard the cardinal number of a class as an indefinable. Hence the above definition avoids a useless indefinable with its attendant primitive propositions, (PM, Vol. II, 4)

while in the "Preface" to Volume I, they write:

> [W]hen what is defined is (as often occurs) something already familiar, such as cardinal or ordinal numbers, the definition contains an analysis of a common idea, and may therefore express a notable advance. . . . In such cases, a definition is a "making definite": it gives definiteness to an idea which had previously been more or less vague. (PM, Vol. I, 12)

Previously, Russell held, in accord with (Aug), that the expression "2", for example, had as its meaning a unique entity, simple or complex, and that that it is "the business of philosophy" to ascertain what that entity is. To paraphrase his pre–Peano comments regarding the decision between that absolute and relative theories of magnitude, "it does not lie with us to choose" whether what that expression means is definable or indefinable; "on the contrary, it is the business of philosophy to discover" what it means. Here, however, Russell and Whitehead present us with a choice as to whether to take numbers to be indefinables, or definable as classes of similar classes, or definable in "some more complicated and practically equivalent definition";[53] and they defend their choice, not on the grounds that it reflects what numerical expressions, as ordinarily used, actually mean, nor, as (PoA) requires, that it reflects what is "present to the mind" when we understand sentences containing numerical expressions, but rather that it enables us to deduce "the formal properties which we expect numbers to have". Compatible with the view that our ordinary notion of cardinal number is "vague", they regard their task not as that of ascertaining "the meaning" of a numerical expression, as it is ordinarily used, but rather as assigning a precise meaning to such an expression that sustains the claims we wish to make regarding numbers.

Similarly, in *Our Knowledge of the External World*, after introducing (Num_{df}), Russell writes:

> This definition . . . yields the usual arithmetical properties of numbers. It is applicable equally to finite and infinite numbers, and it does not require the admission of some new and mysterious set of metaphysical entities. (OKEW, 204)

He then writes in the following paragraph:

> The above definition is sure to produce, at first sight, a feeling of oddity, which is liable to cause a certain dissatisfaction. It defines the number 2, for instance, as the class of all couples, and the number 3 as the class of all triads. This does not *seem* to be what we have hitherto been meaning when we spoke of 2 and 3, though it would be difficult to say *what* we had been meaning. (Ibid.)

And he adds one page later:

> [T]he real desideratum about such a definition as that of number is not that it should represent as nearly as possible the ideas of those who have not gone through the analysis required in order to reach a definition, but that it should give us objects having the requisite properties. Numbers, in fact, must satisfy the formulae of arithmetic; any indubitable set of objects fulfilling this requirement may be called numbers. So far, the simplest set known to fulfill this requirement is the set introduced by the above definition. In comparison with this merit, the question whether the objects to which the definition applies are like or unlike the vague ideas of numbers entertained by those who cannot give a definition, is one of very little importance. (Ibid., 205)

Likewise, in *Introduction to Mathematical Philosophy*, he writes:

> We naturally think that the class of couples (for example) is something different from the number 2. But there is no doubt about the class of couples: it is indubitable and not difficult to define, whereas the number 2 in any other sense, is a metaphysical entity about which we can never feel sure that it exists or that we have tracked it down. It is therefore more prudent to content ourselves with the class of couples, which we are sure of, than to hunt for a problematical number 2 which must always remain elusive. (IMP, 18)

Again, Russell is clear that in identifying cardinal numbers with classes of similar classes, he is not claiming that this captures what we take ourselves to mean by our ordinary statements involving numerical terms; rather, he acknowledges that this account "is sure to produce, at first sight, a feeling of oddity", for what 2 and 3 are on this account "does not *seem* to be what we have hitherto been meaning when we spoke of 2 and 3" and grants, in accord with his earlier view of the cardinal numbers, that we would "naturally" distinguish a cardinal number from a class of similar classes. Again, he does not deny that there are indefinable entities, distinct from classes of similar classes, of the sort he previously took the cardinal numbers to be; rather, he claims only that we "can never feel sure" that there are such "metaphysical entities", so that is "more prudent" to identify the cardinal numbers with classes of similar classes (which are "indubitable") than with such ("problematic") indefinables. What matters is that we find objects that "satisfy the formulae of arithmetic", and since we can be more sure that there are classes of similar classes than that there are those indefinables, it is "safer" to accept his account of the cardinal numbers. [54]

Again, Russell indicates that there is no fact of the matter as to "*what* we had been meaning" in our ordinary use of arithmetical expressions, and that what we entertain when we ordinarily talk of numbers are only "vague ideas". And thus he indicates that the task of analysis is not, as on his Moorean view, a matter of identifying the entity that is "the meaning" a mathematical expression has in its ordinary use, before we undertake the analysis—"the meaning", which by (Aug), such an expression has, in its ordinary use, in order to be meaningful, and which, by (PoA), we must be acquainted with, or have "present to the

mind", in order to understand a sentence containing that expression, as it is ordinarily used—but is rather, as he indicates in PoM, to find a definition that "is at once precise and adequate to all mathematical uses".

Given that Russell's defense of his PoM definitions of cardinal numbers is thus opposed to his earlier Moorean conception of analysis, it is not surprising that Moore himself would object to Russell's definitions. Accordingly, in an unpublished review of PoM, written apparently some time in 1905–6, Moore writes:

> But [Russell's] definition in logical terms of number 'one' is by no means simple: it is as follows: The number 'one' is the class whose members are all those classes, of which each is such has it has a member *x*, such that the proposition "*y* is a member of the class in question and y differs from *x*" is always false, whatever *y* may be. This is the definition in logical terms of the number 'one'. And whether, whenever we say that we have but *one* penny in our pocket, this definition is a correct *analysis* of the property which we mean to attribute to our penny, it is, Mr. Russell admits, permissible to doubt. It is not plain that what we think to be true of the penny, when we think it is but *one*, is no less than that it is a member of the class of [one-membered] classes . . . : it is not plain that this is a correct *analysis* of what we think. That it is *equivalent* to what we think, in the sense that anything whatever which has the property which we mean by 'one' is also a member of this class of classes, and that anything whatever which is a member of this class of classes also has the property which we mean by 'one', there is, indeed, no doubt whatever. But Mr. Russell admits the possibility that it is *only* equivalent—that, possibly, all the members of this class of classes have in common some *other* property, beside the fact that they belong to this class—some other property, which belongs to all of them and only to them, and which may be what we generally mean when we speak of the number 'one'. Mr. Russell, indeed boldly asserts his doubt whether there is any such other property; and there is much to be said for his view.

> But what I wish now to point out is the consequences which follow from the mere possibility that there is such another concept, meant by 'one'. ... When Mr. Russell asserts that $1 + 1 = 2$ can be deduced from logical principles, his assertion only applies to the proposition in which the concept dealt with is 'the class of classes, of which each etc. etc'; it is only *this* proposition which he shows to be deducible from logical principles. If it be true that there is also *another* concept denoted by the word 'one', then the proposition that $1 + 1 =2$, understood as asserting a universal connection between this *other* concept and some others, *cannot* be deduced from logical principles alone. ... Unless, therefore, it can be shown that the concepts dealt with in those propositions, which can be deduced from logical principles, are the very ones which occur in the proposition $1 + 1 =2$, as ordinarily understood, then it must be admitted *either* that the proposition $1 + 1 = 2$, as ordinarily understood, is not a proposition of pure mathematics *or* that Mr. Russell's [logicism] does not ... apply to all propositions of pure mathematics. (G.E. Moore, 1905–6?, 8–10)

Here, Moore is assuming what I have called the "Moorean conception of analysis" to criticize Russell's PoM definition of the number one. Thus he questions whether Russell's "definition is a correct *analysis* of the property which we mean to attribute to our penny" when "we say that we have but *one* penny in our pocket", whether it is "a correct *analysis* of what we think" when we make such a claim or of "what we generally mean when we speak of the number 'one'". Similarly, he questions whether it provides a correct account of the "concepts" which "occur in the proposition $1 + 1 = 2$ as ordinarily understood". And insofar as he suggests that it is implausible to suppose that Russell's definition of the number one—a definition "which is by no means simple"—could be "what we think" or "generally mean when we speak of the number 'one'", then Moore is suggesting, in accord with (PoA), that, if when we make a statement about the number one and what Russell defines the number one to be is not then "before our minds", then Russell's definition cannot be correct.[55]

Further, Moore takes Russell's agnosticism as to whether there are, in addition to classes of similar classes, indefinable properties of the sort Russell previously took numbers to be (and Moore himself, in his 1911 lectures I have quoted above, takes numbers to be) as raising a problem for Russell's logicism. For, given Moore's view, in accord with (Aug), that there is a single entity, which is "the meaning" of the expression "one", as it is ordinarily used, which that expression contributes to propositions such as "$1 + 1 = 2$ as ordinarily understood", then, if what it contributes to that proposition is not the "concept" determined by Russell's definition but rather the indefinable property of one–membered classes which Russell had previously taken the number one to be, then Russell's logicism is threatened. For, if the proposition "$1 + 1 = 2$ as ordinarily understood" contains as constituents indefinable numbers, then it does not consist wholly of constituents definable in purely logical terms, in which case if the proposition "$1 + 1 = 2$ as ordinarily understood" is a proposition of pure mathematics, then Russell's logicism is false. For Moore, as for the pre–Peano Russell, we are not free to introduce definitions as a matter of "convenience"; they have to be answerable to what our words "as ordinarily understood" actually mean. Hence, for Moore, Russell is not free to introduce his definitions of cardinal numbers simply because they facilitate his logicism; if in fact, they do not correspond to "what we generally mean" when we use numerical expressions, then so much the worse for his logicism.

Likewise, given his concern with providing a "phenomenologically" correct analysis of "what we mean" when we make various claims, it is also not surprising that Husserl would reject any analysis that substitutes classes determined by definitions of the form (Ab_{df}) for properties. Accordingly, in his *Logical Investigations*, Husserl criticizes such an analysis, when he writes:

> 'What we mean' is surely our sense, and can one say even for an instant that the sense of the proposition 'This tone is faint' is the same as the sense of the proposition 'This tone belongs to a group (of whatever sort) of similars'?.. .. Naturally the utterances 'A tone is faint' and 'A tone belongs to the sum total of objects alike in their faintness' are semantically equivalent, but equivalence is not identity. (1900–1, Investigation II, Appendix, 303–4)

Here, in accord with some such principle as (PoA), Husserl indicates that since the proposition we entertain when we say "This tone is faint" does not seem to be the same proposition we entertain when we say "This tone belongs to the sum total of objects alike in their faintness", then the proposition expressed by—or "what we mean" when we utter—each of these sentences is not the same (even if they are "equivalent").

Even before he defined cardinal numbers as classes of similar classes, Russell had already introduced definitions that reflect a non–Moorean conception of analysis.

Thus, as I have discussed above, Russell's November 1900 defense of his definition of real numbers as segments of rationals is similar in structure to his later defense of ($\mathrm{Num_{df}}$). In each case, he defends a class–theoretic definition against another account —in the case of real numbers, the accounts of Dedekind, Cantor, and Weierstrass; in the case of cardinal numbers, his own earlier account—according to which the entities in question are taken to be indefinables, whose existence is accepted on the basis of some purported "axiom"— in the case of real numbers, an "axiom of continuity"; in the case of cardinal numbers, his earlier "axiom of abstraction"—or "immediate intuition". In each case, he argues (assuming the existence of classes), that while there can be no doubt of the existence of the classes he has defined, there is less certainty that there are such indefinables. In each case, he argues that since the class–theoretic entities "do all that is required" of the entities in question, there is then no need to assume the alleged indefinables or take them to be the entities in question. In neither case, does he defend his definition on the basis that it reflects "what we ordinarily mean" when we use the term in question.

Similarly, in discussing Cantor's and Dedekind's definitions of "infinite", as well as Cantor's definition of "continuity", which, as I have discussed above, he accepted immediately following the Paris Congress, Russell does not present them as simply articulating "what we mean" when we use these terms. Thus, in "Recent Work on the Principles of Mathematics", he writes "though people had talked glibly about infinity ever since the beginnings of Greek thought", before Dedekind and Cantor "if any philosopher had been asked for a definition of infinity, he might have produced some unintelligible rigmarole, but he would certainly not have been able to give a definition that had any meaning at all". In contrast, "the first and perhaps the greatest step" of Dedekind and Cantor was that they "found . . . a perfectly precise definition of an infinite number or an infinite collection of things" (1901d, 372).

And in discussing, in his November 1900 draft of Part V of PoM, Cantor's definition of "continuity", he writes that "Cantor's merit lies, not in meaning what other people mean, but in telling us what he means himself—an almost unique merit, where continuity is concerned" (PoM, 353).

Thus Russell indicates, in opposition to the Moorean conception of analysis, that the task of defining "infinity" or "continuity" is not to identify "the meaning" of these terms as they are "ordinarily understood", or "what is present to the mind" to someone who understands these terms. Rather, he suggests that prior to Cantor and Dedekind no one, including no philosopher, who had previously used these terms, had any definite meaning "before the mind", and that the achievement of Cantor and Dedekind was to give these terms a precise meaning sufficient for all our theoretical needs.[56] And throughout his writings—including, for example, in PM, Volume I (12), 1919a (105–6), and 1946 (783)—Russell characterizes Cantor's definition of "continuity" as one in which he made "precise" or "definite" what had previously been "vague".

While the post–Peano Russell thus already departs from the Moorean conception of analysis as soon as he introduces his definitions of real numbers and accepts Cantor's and Dedekind's definitions of "infinity" and "continuity" in the period immediately following the Paris Congress, there are some respects in which the departure becomes more dramatic once he accepts ($\mathrm{Num_{df}}$) and other instances of ($\mathrm{Ab_{df}}$). First, one might hold that the notions of "real number", "infinity", and "continuity" are technical or abstruse concepts that, while important for mathematicians and philosophers, are not central to ordinary discourse, in which case few, if any, people understood sentences containing these terms before the appropriate definitions were introduced. However, terms for the cardinal numbers are introduced to children at a very young age and thus seem central our common conceptual repertoire, in which case, it would seem harder to deny that such terms were meaningful or occurred in sentences we were capable of understanding before the definitions were introduced. But in that case, Russell would be forced to conclude, as against (Aug), each such term was meaningful before it succeeded in standing for a unique entity that is its meaning, and, as against (PoA), such a term could occur in a sentence we could understand, even though there was no entity that was "the meaning" of such a term for us to be acquainted with in understanding that sentence.

Further, in the cases of "real number", "infinity", and "continuity", Russell comes to accept definitions of notions he had previously regarded as illegitimate. The pre–Peano Russell had denied that the notion of "the completed or definite infinite" is coherent, and in doing so, he not only rejected Dedekind's and Cantor's definition of "infinity" and Cantor's definition of "continuity", but also denied that there are any real numbers, for he took their existence to depend on the legitimacy of "the completed or definite infinite". In contrast, in the case of cardinal numbers, what changes for Russell is not whether the notion is legitimate or whether there are such entities but rather his view as to what they are. From his later perspective, Russell might claim that his own confusions had prevented him from associating any meaning with "real number", "infinity", or "continuity", so that he was in no position to regard such terms as meaningful or to understand sentences containing them until he overcame these confusions. In contrast, in the case of cardinal number, Russell replaced one definite notion of what they are—indefinables common to similar classes—with another—classes of similar classes. What would seem to be in question here is not whether the term "cardinal number" is legitimate or meaningful at all but rather what meaning to assign to that term. Here, the view of analysis as replacing a "vague" notion with a "precise" one cannot be so easily presented as a case of moving from assigning no meaning to a term and thus having nothing to understand to assigning it a definite meaning and thus having something to understand; rather, it would seem to be a case of choosing among different assignments of meaning to a term that is clearly meaningful and that we understand. Thus, this sort of case presents a more direct threat to his Moorean conception of meaning and understanding. For it suggests, as against (Aug), that an expression can be meaningful without there being a single entity that is its "meaning" and, as against (PoA), that a sentence containing a given expression can be understood without one's having to be acquainted with the entity that is "the meaning" of that expression.

The problem for Russell, however, is that he while never presents his acceptance of ($\mathrm{Num_{df}}$), or other instances of ($\mathrm{Ab_{df}}$), as conforming to

his Moorean conception of analysis, he continues to endorse (Aug) and (PoA), views of meaning and understanding that underpin the Moorean conception of analysis. In particular, by presenting analysis as a process of moving from the "vague" to the "precise", while also officially accepting (Aug) and (PoA), Russell is committed to holding that our words — including, commonly used numerical expressions—that, as ordinarily used, are "vague" are meaningless and cannot occur in sentences that we understand.

I argue now that once he accepts his behaviorist account of meaning and understanding in his post post–1918 writings, Russell is able to avoid this conclusion; for by introducing his behaviorist account of meaning and understanding, Russell is in a position to reject (Aug) and (PoA), thereby it possible for him to hold that "vague" language can be meaningful and understood and thereby making more coherent and plausible his post–Peano practice of analysis.

3.3. Vagueness, Analysis, and Behaviorism

The topic of vagueness plays a central role in a number of Russell's post–1918 publications, including, most prominently, his influential 1923 paper "Vagueness".[57] Those who have considered the genesis of Russell's interest in the topic have typically found little, if any, connection between his post–1918 discussions of vagueness and his earlier philosophy of mathematics in either PoM or PM.[58] As against this, I argue that the definition of vagueness Russell provides in his post–1918 writings captures the notion of vagueness he appeals to in defending ($\mathrm{Num_{df}}$); however, I argue also that what Russell possesses in his post–1918 writings that he previously lacked is a theoretical framework that allows vague language to be meaningful and capable of being understood, thereby enabling him engaging in his post–Peano practice of analysis without having to regard language, as it is ordinarily used, prior to analysis as meaningless and not capable of being understood. While Russell's post–Peano practice of analysis leads him to recognize the phenomenon of vagueness that he later articulates more fully, it is not until he adopts a behaviorist account of meaning and understanding that he has a plausible account of that phenomenon and hence of that practice of analysis.

In "Vagueness", Russell holds that "vagueness and precision alike are characteristics which can only belong to a representation, of which language is an example” (1923a, 147). He holds generally that a representational system is “precise” when there is a one–one relation between that system and the system it represents, while “a representation is *vague* when the relation of the representing system to the represented system is not one–one but one–many” (ibid., 152). For Russell, “a photograph which is so smudged that it might equally represent Brown or Jones or Robinson is vague”, and a map is more vague to the extent that “various different courses [of, say, a road or river] are compatible with the representation it gives” (ibid.). Then “passing from representation in general to the kinds of representation that are specially interesting to the logician”, Russell writes:

> . . . the representing system will consist of words, perceptions, thoughts, or something of the kind, and the would–be one–one relation between the representing system and the represented system will be *meaning*. In an accurate language, meaning would be a one–one relation; no word would have two meanings, and no two words would have the same meaning. In actual languages . . . meaning is one–many. . . . That is to say, there is not only one object that word means, and not only one possible fact that will verify a proposition. The fact that meaning is a one–many relation is the precise statement of the fact that all language is more or less vague. (Ibid.)

In indicating that in vague language, “meaning” is a “one–many relation”, Russell is not, I take it, claiming that a vague word succeeds in standing for more than one entity; instead, he is indicating that a vague word fails to succeed in picking out, or standing for, one and only one entity as its meaning. That is, a vague word is such that it is compatible with the use of that word that different entities be taken as “the entity” that it stands for, in which case, there is nothing in the use of that word which singles out one of those entities, to the exclusion of the others, as “the entity” it stands for;[59] but this is not to say that one is entitled to take that vague word as standing for all those entities together. Thus, in Russell’s example, the smudged photograph that is vague “might equally represent Brown or Jones or Robinson”; but this is not to say that it thereby represents all of them simultaneously. Rather, there is only one person that the photograph can be of; but there is

nothing in that photograph that determines which person that is.

Again, Russell argues that because “birth is a gradual process” and likewise death, the name “Ebenezer Wilkes Smith” is vague, for it does not succeed in designating one and only one particular whose existence begins and ends at determinate times (ibid., 149). Given that “Ebenezer Wilkes Smith” is a proper name, we take it as referring to only one entity; but there is nothing in our use of that name that that enables it to pick out exactly which entity, with exactly which life–span, that is. Likewise, for Russell, because “there are shades of colour concerning which” it is “essentially doubtful” whether the word “red” applies to them or not, the predicate “red” is vague (ibid., 148); for Russell, since “red” is a predicate, we take it to stand for one universal, but there is nothing in our use of that word determining exactly which universal, encompassing which shades of color, that is.[60]

This account of vagueness is in accord with Russell’s PoM characterization of numerical expressions as vague. For by indicating there that so far as “all mathematical uses” of numerical expressions goes, we can take those expressions to be referring either to the indefinables that he had previously taken the cardinal numbers to be or to be classes of similar classes, Russell is indicating that there is nothing in our ordinary use of such expressions that determines which (if either) of these sorts of entities those expressions stand for—which is not to say that such expressions stand for both sorts of entities. For while we may interpret such an expression as standing for either sort of entity, we cannot interpret it as standing for both sorts of entity simultaneously. Thus, for Russell (in PoM[61]), while we take a numerical expression to refer to a single entity, there is nothing in our ordinary use of that expression which determines exactly which entity that is, in which case such an expression fails to succeed in standing for one entity to the exclusion of others. Hence, the phenomenon of vagueness that Russell identifies in arithmetical discourse in 1903 is the same phenomenon he characterizes generally in his 1923 paper.

What has changed by 1923 is that Russell now has a general account of meaning and understanding that puts him in a position to hold that vague discourse may be meaningful and understood. So long as he officially accepts (Aug) and (PoA), he is no position to hold that a vague expression—that is, an expression which does not succeed in standing for one and only one entity—is meaningful or can be understood. However, once he holds that

> A person "understands" a word when (a) suitable circumstances make him use it, (b) the hearing of it causes suitable behavior in him, (1919b, 290; 1921, 197)

and that:

> Understanding words does not consist in knowing their dictionary definitions, or in being able to specify the objects to which they are appropriate. ... Understanding language is more like understanding cricket: it is a matter of habits, acquired in oneself and rightly presumed in others. To say that a word has a meaning is not to say that those who use the word correctly have ever thought out what the meaning is: the use of the word comes first, and the meaning is to be distilled out of it by observation and analysis, (1921, 197; see also 1919b, 290)

then there is room for the possibility that word can be meaningful and understood, even if it is vague. For since what is now required for a word to be meaningful is for it to be used in certain ways, it becomes an open question whether the use which is sufficient for establishing that a given word is meaningful is also sufficient for determining a unique entity that is "the meaning" of that word. If it is not thus sufficient, then, as against (Aug), that word is meaningful, even though it is vague. Likewise, if understanding that word is simply a matter of using and reacting to it in certain ways, and if the use which is sufficient for establishing that someone has understood that word is not sufficient for regarding that person as having identified a unique entity, to the exclusion of others, as "the meaning" of that word, then, as against (PoA), a person may understand that word without being acquainted with any entity that is "the meaning" of that word.

This is not to say that everyone who adopts such a behaviorist account of meaning and understanding will automatically deny that any, let alone each, word fails to succeed in standing for one and only one entity that is thereby "the meaning" of that word. For some might hold that the behavior which is sufficient for establishing that a word is

meaningful is also sufficient for identifying a unique entity that is "the meaning" of that word. This, however, is not Russell's view.[62] For immediately after the passage from *The Analysis of Mind* I have just quoted, he continues:

> Moreover, the meaning of a word is not absolutely definite: there is always a greater or lesser degree of vagueness. The meaning is an area like a target: it may have a bull's eye, but the outlying parts of the target are still more or less within the meaning, in a gradually diminishing degree as we travel further from the bull's eye. As language grows more precise, there is less and less of the target outside the bull's eye, and the bull's eye itself grows smaller and smaller; but the bull's eye never shrinks to a point, and there is always a doubtful region, however small, surrounding it. (1921, 197–8)

Thus, for Russell, whatever we can "distill out of" the use of a given word, it will not be enough to pick out one unique entity that is "the meaning" of that word. For Russell, once we cease to assume at the outset, as (Aug) requires, that for a given word to be meaningful is for it to stand for one and only one entity that is its "meaning", and instead hold that for a given word to be meaningful is for it to be used in certain ways, we will be able to raise the question whether a meaningful word succeeds in standing for a unique entity that is its "meaning"; and once we raise that question, the answer we should give is that no meaningful word ever thus succeeds. Here, I am not concerned to assess Russell's view that once meaning is understood behaviorally, we will not, in general, be able to identify a unique entity that is "the meaning" of a given word; rather, I am only concerned to point out how, by accepting this view, he is able to reject (Aug) and (PoA), and thereby hold that vague language may be meaningful and understood.

In coming to hold this view, Russell's theory has finally caught up with his practice. The Moorean conception of analysis presumes that words as we ordinarily use them are meaningful and can be understood by standards of (Aug) and (PoA). Thus each word succeeds in standing for a unique entity; each sentence succeeds in expressing a unique proposition; and the task of analysis is to identify the constituents of the proposition expressed by a given sentence—that is, the meanings of the words in that sentence with which, by (PoA), we must be acquainted in order to understand that sentence. In contrast, Russell's post–Peano practice of analysis of mathematical discourse begins with the use of certain expressions—for example, the numerical expressions in "the formulae of arithmetic". There is no assumption that each word in the relevant sentences succeeds in standing for a unique entity to the exclusion of others, but only the assumption that the relevant sentences are true; here, the task of analysis is not to identify "the" constituents of "the" propositions expressed by the sentences in question but rather to specify propositions and thus to specify meanings for the words in those sentences that will insure that the sentences which we take at the outset to be true are, in fact, true. Here, when we begin with sentence S_1 and end up with sentence S_2 as the outcome of our analysis, we do not regard S_2 as a perspicuous representation of "the" proposition we originally apprehended when we originally understood S_1; rather, we regard S_1 as "vague", as not succeeding in expressing one proposition to the exclusion of others, and we regard S_2 as precisely (or at least more precisely) expressing a proposition that we are confident will have the same truth–value we wish to assign to S_1, but a proposition one need not apprehend when one originally understood S_1.

Insofar as Russell holds that the sentences we take to be obviously true at the outset of our analysis are meaningful, his post–Peano practice of analysis thus requires is a view of meaning that enables us to ask how we may interpret an expression that is meaningful but vague.; and this is what he provides in his post–1918 view that "the use of the word comes first, and the meaning is to be distilled out of it by observation and analysis". Use is what determines whether an expression is meaningful, and use is the condition that any interpretation of an expression must meet; where its use is compatible with interpreting that expression in more than one way, then that expression is vague; and in such a case, the task of analysis is to find a "prudent" interpretation of that expression compatible with its use—in particular, an assignment of meaning that will "safely" allow us to assign the truth–values we wish to assign to the sentences containing the expression in question. And it is in these sorts of terms that Russell typically characterizes the task of analysis in his post–Peano writings.

Thus, for example, in his 1914 paper, "The Relation of Sense–Data to Physics", Russell writes that "the supreme maxim in scientific philosophizing is this: *Wherever possible, logical constructions are to be substituted for inferred entities*" (1914a, 11), and, as examples of these, he mentions his PoM definition of the irrationals, his PoM definition of the cardinal numbers, and his PM "no classes" account of discourse apparently about classes. In these cases, Russell provides definitions that enable us to prove propositions in the relevant domain without having to hold that there are indefinables of the sort that some (Dedekind, Cantor, and Weierstrass, in the case of the irrationals; his own earlier self, in the case of the cardinals; and Frege along with his earlier self, in the case of classes) have taken irrationals, cardinal numbers, and classes to be, and without having to accept purported "axioms" (such as Dedekind's "axiom of continuity", or his own earlier "axiom of abstraction", or Frege's Axiom V) that would guarantee the existence of such indefinables. In none of these cases, does Russell actually deny that there are such indefinables; and in none of them does Russell proceed in accord with the Moorean conception of analysis. For, Russell's central claim is not that his definitions reflect "what we mean" or what is "present to our minds" when we understand sentences containing the defined expressions but rather that given his definitions, the sentences we take to be true in the relevant domains will be true whether or not there are such indefinables.

I have made these points above with regard to his PoM definitions of the irrationals and the cardinal numbers; and they hold as well for his "no classes" theory, which he introduces in PM by writing:

> It is not necessary for our purposes ... to assert dogmatically that there are no such things as classes. It is only necessary for us to show that the incomplete symbols which we introduce as representatives of classes yield all the propositions for the sake of which classes might be thought essential. (PM, Vol. I, 72)

Again, Russell presents himself as developing a theory that "yield[s] all the propositions for the sake of which classes might be thought essential" without having to assume that there are "such things as classes". He does not "assert dogmatically that there are no such things as classes"; nor does he present himself as providing an account of "what we mean" or of what is "present to our minds" when we understand "the propositions for the sake of which classes might be thought essential". Rather, he is seeking to arrive at an interpretation of sentences containing class symbols that enables us to assign the truth–values we take those sentences as having, irrespective of how far that interpretation may be from what was "present to our minds" when we first understood such sentences. Just as his class theoretic definitions—or "logical constructions", in his terminology of 1914—of the irrationals and cardinal numbers will insure that the relevant formulae are true regardless of whether there are indefinables—"inferred entities", in his 1914 terminology, whose existence follows from the supposed relevant "axioms"—of the sort others have taken irrationals and cardinal numbers to be, so too the "logical constructions" of his "no class" theory will "yield all the propositions for the sake of which classes might be thought essential" regardless of whether there actually are the "inferred" entities—namely, classes themselves—that people have thought those propositions are about.

Accordingly, after mentioning these three cases in which he had "substituted logical constructions" for "inferred entities", Russell continues in "The Relation of Sense–Data to Physics" by writing:

> The method by which the construction proceeds is closely analogous in these and all similar cases. Given a set of propositions nominally dealing with the supposed inferred entities, we observe the properties which are required of the supposed entities in order to make these propositions true. By dint of a little logical ingenuity, we then construct some logical function of less hypothetical entities which has the requisite properties. This constructed function we substitute for the supposed inferred entities, and thereby obtain a new and less doubtful interpretation of the body of propositions in question. This method, so fruitful in the philosophy of mathematics, will be found equally applicable in the philosophy of physics, where, I do not doubt, it would have been applied long ago but for the fact that all who have studied this subject hitherto have been completely ignorant of mathematical logic. (1914a, 12)

The "method" which Russell describes here and seeks to apply to statements of physics is not that which would be required by a Moorean conception of analysis. Rather than presenting himself as identifying the constituents of the propositions expressed by certain sentences—and

hence, by the (PoA), as identifying the entities with which one must be acquainted in order to understand those sentences—Russell indicates that his purpose is arrive at an "interpretation" of a given body of sentences that will insure that those sentences are true. Thus, the "data" Russell is seeking to accommodate by this method are not Moorean "data" as to "what we mean" or what is "before the mind" when we understand those sentences, but rather the "data" that those sentences are true; and thus Russell will succeed in his task so long as the "interpretation" he provides of those sentences gives them the truth–values we pre–theoretically take them as having, regardless of whether it also gives them the "meaning" we pre–theoretically take them as having.[63]

In the lecture series "The Philosophy of Logical Atomism" (PLA) he delivered in January–March 1918, shortly before he went to prison that May, Russell similarly describes his conception of analysis, this time appealing to the notion of "vagueness". Thus, early in the first of those lectures, Russell remarks:

> It is a rather curious fact in philosophy that the data which are undeniable to start with are always rather vague and ambiguous. You can, for instance, say: "There are a number of people in this room at this moment." That is obviously in some sense undeniable. But when you come to try and define what this room is, and what it is for a person to be in a room, and how you are going to distinguish one person from another, and so forth, you find that what you have said is most fearfully vague and that you really do not know what you meant. That is a rather singular fact, that every thing you are really sure of, right off is something you do not know the meaning of, and the moment you get a precise statement you will not be sure whether it is true or false, at least right off. The process of sound philosophizing, to my mind, consists mainly in passing from those obvious, vague, ambiguous things, that we feel quite sure of, to something precise, clear, definite, which by reflection and analysis we find is involved in the vague thing that we start from, and is, so to speak, the real truth of which that vague thing is a shadow. ... Everything is vague to a degree you do not realize till you have tried to make it precise,

> and everything precise is so remote from everything that we normally think, that you cannot for a moment suppose that is what we really mean when we say what we think. (PLA, 161–2)

Here, Russell presents a view of vagueness and of analysis which is in accord with both his post–Peano "analyses" of mathematical concepts[64] and his post–1918 characterization of vagueness, but which is opposed to his Moorean views of analysis and meaning. Thus, in claiming that although "the data" with which we begin in philosophy are claims we take to be obviously true, those claims themselves are "fearfully vague", so that "you really do not know what you meant" in making such claims, Russell is characterizing his post–Peano procedure of beginning with statements in a given domain that we take to be true and then attempting to find a precise "interpretation" of those statements which, while insuring that those statements are true, is not intended to reflect "what we really mean" when we originally made those statements. And while that procedure is in accord with his post–1918 view that "the use of the word comes first" and that "the meaning" of a word that may be "distilled out" of its use is "not absolutely definite", it is opposed to the Moorean conception of analysis, according to which, given (PoA), it is incoherent to suppose that we can understand a sentence and take it to be obviously true, and not know "what we mean" in uttering that sentence.

Russell's notion of vagueness is not only central to his post–Peano characterization of analysis as "passing" from what is "obvious" and "vague" to "something precise, clear, definite, which by reflection and analysis we find is involved in the vague thing that we start from"; it also sets the main problem that his methodology of "substituting" "logical constructions" for "inferred entities" is meant to address. For given that a vague claim is one that admits of many different precise interpretations, then insofar as the original vague claim will be true so long as any of those precise interpretations is true, it is more likely that the original claim will be true than will any of its precise interpretations. Accordingly, in "Vagueness", Russell writes: "A vague belief has a much better chance of being true than a precise one, because there are more possible facts that would verify it" (1923a, 153; see also 1921, 188). Similarly, in the passage I have just quoted from PLA, Russell contrasts

the certainty of the vague ("everything you are really sure of, right off is something you do not know the meaning of") with the lack of certainty of the precise ("the moment you get a precise statement you will not be sure whether it is true or false, at least right off"); and in the following paragraph he adds:

> When you pass from the vague to the precise by the method of analysis and reflection that I am speaking of, you always run a certain risk of error. If I start with a statement that there are so and so many people in this room, and then set to work to make the statement precise, I shall run a great many risks and it will be extremely likely that any precise statement I make will be something not true at all. So you cannot very easily or simply get from these vague undeniable things to precise things which are going to retain the undeniability of the starting–point. (PLA, 162)

Accordingly, the challenge of analysis for Russell is to move from a vague statement we take to be obviously true to a precise interpretation of that statement that is less at risk of being false than are other precise interpretations of that statement. And accordingly also, Russell's main defense of his "logical constructions" is that they provide interpretations of the original "obvious" statements that render it more certain that those original statements are true than do interpretations according to which those statements are about "inferred" entities. Thus, in PoM, Russell argues that whether or not there are indefinables of the sort he had previously taken the cardinal numbers to be, there are at least classes of similar classes, in which case interpreting "the formulae of arithmetic" as being about classes of similar classes renders it more certain that those formulae are true than interpreting them as being about the former sort of indefinables.

In the final lecture of PLA, Russell presents the task of analysis in similar terms. In particular, he discusses "the purpose embodied in the maxim called Occam's Razor" as follows:

> [T]ake some science, say physics. You have there a given body of doctrine, a set of propositions expressed in symbols . . . and you think that you have reason to believe that on the whole those propositions, rightly interpreted, are fairly

> true, but you do not know what is the actual meaning of the symbols that you are using. The meaning they have *in use* would have to be explained in some pragmatic way: they have a certain kind of practical or emotional significance to you which is a datum, but the logical significance is not a datum, but a thing to be sought, and you go through, if you are analyzing a science like physics, these propositions with a view to finding out what is the smallest empirical apparatus—or the smallest apparatus, not necessarily wholly empirical—out of which you can build up these propositions. (Ibid., 235)

And he adds later:

> [E]very diminution of apparatus diminishes the risk of error. Suppose, e.g., that you have constructed your physics with a certain number of entities and a certain number of premisses; suppose you discover that by a little ingenuity you can dispense with half those entities and half of those premisses, you clearly have diminished the risk of error, because if you had before 10 entities and 10 premisses, then the 5 you have now would be all right, but it is not true conversely that if the 5 you have now are all right, the 10 must have been. Therefore you diminish the risk of error with every diminution of entities and premisses. (Ibid., 242–3)

Moreover, with regard to the entities he thus "dispenses" with, he claims: "I am not *denying* the existence of anything; I am only refusing to affirm it." (Ibid., 237) Thus, for example, in discussing his view that we can dispense with "the metaphysical and constant desk" and treating it as a "logical construction" out of "appearances", he writes:

> You have anyhow the successive appearances, and if you can get on without assuming the metaphysical and constant desk, you have a smaller risk of error than you had before. You would not necessarily have a smaller risk of error if you were tied down to *denying* the metaphysical desk. That is the advantage of Occam's Razor, that it diminishes your risk of error. (Ibid., 243)

Again, Russell is here describing his post–Peano practice of analysis: begin with a body of propositions we take to be true but with which we have associated no precise meaning, and find a precise interpretation of those propositions that minimizes the "risk" of those propositions being false. In doing so, we will "dispense" with certain entities we may have formerly assumed to exist, but in thus dispensing with those entities, we are not positively denying that they exist, only refusing to affirm that they do. For we increase our "risk of error"—we make an added claim on reality that might be false—if we either affirm or deny that they exist. Again, the procedure he describes is opposed to his Moorean conception of analysis, which precludes the possibility that a sentence can have a meaning "in use" for us without it expressing a definite proposition that we apprehend, and which takes the task of analysis to consist in identifying "the" constituents of "the" proposition we thus apprehend in understanding the sentence in question, not in finding an "interpretation" of that sentence that minimizes the "risk" of that sentence's being false.

As late as PLA, however, the problem remains for Russell that insofar as he holds that "vague undeniable" statements which we accept at the outset of our analyses are meaningful and can be understood, then his post–Peano practice of analysis, which he accurately describes in both the first and last lectures in that series, is incompatible with views of meaning and understanding, including (Aug) and (PoA), he officially endorses (especially in the context of presenting his most well–known analysis, namely the theory of descriptions[65]), including in that lecture series. Thus, in the second lecture he says: "When I speak of a symbol I simply mean something that 'means' something else"(ibid., 167); and in the third he writes: "[T]o understand a symbol is to know what it stands for"(ibid., 182). Further, he claims that "the meaning you attach to your words must depend on the nature of the objects you are acquainted with" (ibid., 174). More specifically, he defines proper names as words that "stand for" particulars (ibid., 178), while "by a predicate ... I mean a word that is used to designate a quality such as red, white, square, round" (ibid., 182). He says also that "to understand a name, you must be acquainted with the particular of which it is a name" (ibid.), while in discussing what is involved in understanding the word "red", he writes:

> [S]uppose,—as one always has to do—that "red" stands for a particular shade of colour. You will pardon that assumption, but one never can get on otherwise. You cannot understand the word "red" except by seeing red things. There is no other way it can be done. . . . All analysis is possible in regard to what is complex, and it always depends in the last analysis, upon acquaintance with the objects that are the meanings of certain simple symbols. . . . In the sense of analysis, you cannot define "red". (Ibid., 173–4)

All these remarks are in accord with views Russell accepted during his Moorean period: the meaning of a word (at least, of a name or a predicate) is a unique entity that that word "stands for"; understanding a word (at least, a name or predicate) requires knowing, or being acquainted with, the entity it stands for; where a word stands for a simple entity, that word cannot be defined or understood in terms of other expressions already understood, but can only be understood by acquaintance with the simple entity that its meaning. While he hints here at the vagueness of the predicate "red", he indicates that "one never can get on" without assuming that it "stands for a particular shade of meaning". Whereas in the first lecture, he presents vagueness as central to his whole conception of analysis, here he presents vagueness as a phenomenon that has to be ignored if we are to characterize meaning and understanding. Whereas in his final lecture, he alludes to a notion of "meaning" an expression can have "in use", which does not require us to "know" any definite meaning corresponding to that expression, in these passages, in accord with (Aug) and (PoA), he identifies having meaning with standing for a unique entity, and understanding an expression with being acquainted with the entity it stands for.

By "On Propositions", written in February and March 1919, Russell finally gives up these Moorean views of meaning and understanding. No longer is it necessary for a word to be meaningful that succeed in standing for a unique entity that is "the meaning" of that word; no longer is it necessary for "understanding" a word that a person should "know what it means"; no longer is it necessary for Russell to ignore the phenomenon of vagueness when presenting his official views of meaning and understanding. In the first lecture in PLA, in the course of characterizing "the method of analysis" as "pass[ing] from the vague to the precise", Russell remarks: "I should like, if time were longer and if I knew more than I do, to spend a whole lecture on the conception of vagueness." (Ibid., 161) Given that accepting the conception of vagueness central to his post–Peano practice of analysis requires rejecting (Aug) and (PoA)—principles he continues to advocate in PLA—if he is to allow that vague discourse can be meaningful and understood, then it is perhaps not surprising that in PLA, he is not yet in a position to deliver "a whole lecture on the conception of vagueness". By adopting a behaviorist account of meaning and understanding shortly after PLA, enabling him to reject (Aug) and (PoA), Russell is finally in a position to provide a coherent account of vagueness and hence of the "method of analysis" he has practiced since shortly after the Paris Congress.

In *Word and Object*, in discussing the set–theoretical definition of "ordered–pair", Quine writes:

> This construction [of the ordered pair] is paradigmatic of what we are most typically up to when in a philosophical spirit we offer an "analysis" or "explication" of some hitherto inadequately formulated "idea" or expression. We do not claim synonymy. We do not claim to make clear and explicit what the users of the unclear expression had unconsciously in mind all along. We do not expose hidden meanings, as the words 'analysis' and 'explication' would suggest; we supply lacks. We fix on the particular functions of the unclear expression that make it worth troubling about, and then devise a substitute, clear and couched in terms to our liking that fills those functions. (1960, 258–9)

Quine's account here is in accord with Russell's post–Peano view of analysis as "pass[ing] from the vague to the precise". Like Russell, and as against (Aug) and (PoA), Quine denies that there was one definite meaning that that phrase had prior to the analysis and that understanding that phrase was a matter of having such a meaning "in mind". Hence, like Russell, Quine does not present himself as articulating "the" meaning that the phrase to be analyzed had prior to the analysis or the meaning that the users of that phrase "had unconsciously in mind all along". Like Russell, Quine presents the analysis as a matter of replacing an unclear expression which fulfilled certain functions by a clear expression that fulfills those same functions. [66]

However, when Quine presents this account of analysis in Chapter 5 of *Word and Object*, he had already argued in Chapter 2, on the basis of a behaviorist view of language, against the view that each term succeeds in standing for a unique entity that is "the meaning" of that term or that each sentence succeeds in expressing a unique proposition, and thus against a view of language that would sustain the Moorean conception of analysis. In Russell's own development, in contrast, while his post–Peano practice of analysis presumes that terms to be analyzed do not succeed in having unique "meanings" and that the process of analysis is one of "passing from the vague to the precise", it only after 1918 that he arrives at a behaviorist view of language that enables to make coherent that practice of analysis.[67] Whereas Quine presents his view of analysis as following from his earlier rejection of synonymy and determinate meaning, for Russell, the practice of non–Moorean analysis precedes by more than fifteen years the theory of language that he eventually uses to underpin it.

CONCLUSION

As I have discussed at the outset, recent and growing interest in Russell's post–1918 publications calls into question the view that, throughout his philosophical development, Russell adhered to a number of views—including a foundationalist epistemology, (Aug), and (PoA)—that were only challenged within the analytic tradition by such philosophers as the later Wittgenstein, Quine, and Sellars in the middle part of the twentieth century; for it is clear that by the early 1920's Russell himself had already rejected those views. Here, I have argued further that just as it would be wrong to ignore his post–1918 publications when considering Russell's place in the analytic tradition, so too it would be wrong to acknowledge how opposed those post–1918 publications are to the stereotyped view of Russell's philosophy while also presenting those post–1918 publications as expressing views for which there is no precedent in his earlier writings. More specifically, I have argued that just as it would be wrong to suppose that after he broke with Idealism, Russell was throughout the rest of his philosophical development a foundationalist epistemologist who adhered to (Aug) along with (PoA), it would also be wrong to suppose that in his writings through 1918

at least, Russell was a foundationalist epistemologist whose practice of analysis uniformly reflects a commitment to (Aug) along with (PoA).

As I have presented Russell's development, following the Paris Congress through his writings in 1918, there is a tension in Russell's philosophy between the Moorean philosopher, who *is* a foundationalist epistemologist who accepts (Aug) along with (PoA), and the post–Peano philosopher, who undermines central Moorean principles. While the Moorean principles are central to the broad ideology for much of Russell's general anti–Idealist rhetoric, the post–Peano philosopher is one whose general philosophical views emerge as a result of technical commitments. Thus, the post–Peano philosopher does not set out to argue on general epistemological grounds against foundationalism; however, the details of accepting Cantor's theory of the transfinite and, later, of responding to the paradoxes commit Russell to views at odds with his Moorean foundationalism. Likewise, the post–Peano philosopher does not set out to present an account of meaning and understanding that is opposed to (Aug) and (PoA); however, by introducing his definitions of irrationals and cardinal numbers, by later developing his "no classes" theory, and then by applying the techniques he developed in the philosophy of mathematics to other areas, including the philosophy of physics and the philosophy of mind, Russell becomes committed to a view of analysis that is opposed not only to the Moorean conception of analysis but also to (Aug) and (PoA), which underpin the Moorean conception of analysis.

On this interpretation, what Russell achieves by the 1920's is a general philosophical outlook that is clearly aligned with the commitments and practice of the post–Peano philosophical and that is thereby opposed to fundamental features of the Moorean philosophy.[68] Already, by the 1910's, and in accord with anti–foundationalism required to accept Cantor's theory and to address the paradoxes, Russell recognizes no sharp distinction between philosophy and science; but by accepting further, by the 1920's, a behaviorist account of meaning and understanding, to which he was led by reading Watson but which enabled him to reject (Aug) and (PoA) and account for the phenomenon of vagueness that is central to his post–Peano practice of analysis, Russell's naturalism—and disdain for traditional philosophy—becomes more pronounced. Freed of the basic epistemological and semantic commitments of his earlier Moorean philosophy, Russell now practices a philosophy that sets him apart, in both substance and style, from any other philosopher in the analytic tradition until the emergence of Quine.[69]

Notes

[1] See, for example, Pears (1967), Ayer (1971), and Sainsbury (1979).

[2] For other recent work emphasizing Russell's naturalism, see, for example, O'Grady (1995), Lugg (2006a,b), Stevens (2005, 2006a,b), and Kitchener (2007).

[3] I recognize that Russell often presents himself as differing with behaviorists in that he recognizes the existence of mental images (see, for example, 1919b, 283–90); however, as Dreben points out (1996, 59, note 52), by 1926 Russell endorses the view that "'images' should not be used in explaining 'meaning'" (1926, 140).

[4] However, they do not include all features of the stereotyped picture—in particular, the view that we are acquainted with sense–data, but not with material objects.

[5] For this sort of characterization of Absolute Idealism, see, for example, Russell (1899–1900, 39, 96, 160–1).

[6] During his Moorean period, Russell regards numbers as properties of plural subjects (see, for example, PoL, 12). This is opposed to the Fregean view Russell later came to accept that "statements of number" are about properties, not objects (see, for example, OKEW, 201–2). See Byrd (1987, 65–6) for some discussion of how this change in view is reflected in late additions Russell made to PoM.

[7] For early statements by Moore of the "act–object" distinction, see, for example, his 1902, 157; see also the discussion by Baldwin (1990, 12–16).

[8] After the Paris Congress, Russell rewrote 1901b in English and had it published in *Mind* as 1901c, making numerous changes, largely as a result of discussion with Moore, to the first part of the paper, which concerns the detail of his argument regarding time and space (see CP3, 259–60). However, he also inserts the following into the passage I have quoted in which he criticizes Lotze:

> In short, all knowledge must be recognition, on pain of being mere delusion; Arithmetic must be discovered in just the same sense in which Columbus discovered the West Indies, and we no more create numbers than he created the Indians. The number 2 is not purely mental, but is an entity which may be thought of. (1901c, 278; see also PoM, 451)

Here, by using the word "knowledge", Russell clearly associates our thinking *of* an entity with our "knowing" or "recognizing" that entity; in doing so, he is, I take it, simply articulating more explicitly what had been his Moorean position all along.

[9] As he does in PoM, "Preface", third–to–last paragraph.

[10] One reason Russell comes to emphasize (PoA) more in his later writings is (as Hylton has emphasized (1990, 245–8); see also my 1998a, 416–9, 440) that he increasingly recognizes limitations on our powers of acquaintance. Thus, for example, in PoP he argues that we are not acquainted with physical objects in which case, by (PoA), they cannot be constituents of propositions we can apprehend. Earlier in PoM (145), he argues that since we (humans) cannot apprehend an infinitely complex proposition, then, by (PoA), infinite classes (which, following the Paris Congress, he recognizes as completed totalities—see Part 2 below) cannot be constituents of propositions we can apprehend. During his Moorean period, however, Russell seems to recognize no limitations on our powers of acquaintance—each entity can be an "object" of one of our ideas. Hence, during his Moorean period, Russell has no occasion to use the (PoA), as he does later, as the engine driving the analysis of propositions expressed by sentences which we understand but which contain expressions that seem to stand for entities with which we are not acquainted. In these sorts of cases, it becomes, in Russell's words I have quoted above, "the fundamental principal in the analysis of propositions containing descriptions" (1911b, 154; PoP, 58).

[11] I take it that, for Russell, since his concern here is the "objects" of our mental acts, not those mental acts themselves, then what is "present" to, or "before", our minds, and not what is "in" our minds, is a more precise statement of his real concern here, something he does not make clear in this relatively popular essay.

[12] Russell also writes: "In cases where, as with numbers and colours, these positions [that is, the absolute positions in independent series] have names, the absolute theory is plainly correct." (1900b, 226) Here, Russell seems to be assuming some such view as that, consistent with (PoA), in understanding a name of number or color, we will thereby know that it stands for an indefinable ultimate constituent of the universe.

[13] This argument is central to Russell's initial rejection of Idealism; for some discussion of it, see Griffin (1991, Chapter 8, section 6) and also my 1998b, 103–7.

[14] Thus the defense of "common sense" against philosophical theorizing—prominent Moore's later writings, such as "A Defense of Common Sense" (1925) and "Proof of an External World" (1939)—is also reflected in Russell's early Moorean philosophy.

[15] For an overview of some of these issues, see Grattan-Guinness (1996–7).

[16] For some discussion of these points, see G. H. Moore's "Introduction" to CP3 (xxvi–xxvii) and Byrd (1994, 56–64); see also below Part 3.

[17] See, for example, Moore & Garciadiego (1981, 328–9).

[18] Thus, I am not here concerned with the issue as to what specific features of Peano's own characteristic views Russell came to accept after the Paris Congress. This is the issue Russell himself focuses on in discussing "the enlightenment that I derived from Peano" (1959, 66).

[19] For a fuller discussion of which, see G. H. Moore (1995, 226f).

[20] An assumption which, Russell came to realize, applies, on Cantor's theory, to transfinite ordinal numbers, but not to transfinite cardinals.

[21] Convergent in the sense that the difference between consecutive members of that sequence becomes as small as we like, if they are sufficiently far out in the series (see PoM, 281, for this sense of convergence).

[22] Russell's criticism here of Dedekind is the basis for his later remark that Dedekind's "method of postulating" the irrational numbers as the limits of certain sequences of rational numbers "has all "the advantages of theft over honest toil" (1919a, 71).

[23] One may ascertain which passages from Parts III–V PoM were in Russell's November 1900 draft by comparing PoM with "the list of variants" between the draft and published text compiled by Byrd (1994, 1996–7). Where there is such a change in a passage quoted, my citation for the passage mentions both PoM and Byrd, and I have indicated in brackets how the published text differs from the November 1900 draft. Where there is no such change, I have simply cited PoM.

[24] Thus, for Russell, the class of all rationals less than 1/2 is the real number 1/2

(which he thereby distinguishes from rational number 1/2—see PoM, 270); and the class of rational numbers which are such that their squares are less than 2 is the real number $\sqrt{2}$.

[25] Sometime between June 1902 and February 1903 while he corrects the final proofs of the book, Russell acknowledges the problems raised by his paradox, by adding "although certain special infinite classes do give rise to hitherto unsolved contradictions" (PoM, 368, as collated with Byrd, 1994, 86).

[26] For some sense of Russell's influence in shaping how the history of mathematics has been understood, see the references to Russell in such books as Boyer (1949, e.g., 3, 13, 25, 293) and Foster Wallace (2003, e.g., 48, 52, 180–1, 198, 258). Ehrlich comments that Russell's PoM and 1901d are "the works which perhaps more than any other helped to promulgate . . . among historians and philosophers of mathematics" the views that "the nature of the infinite and the continuum were completely revealed by Cantor and Dedekind, and the concept of an infinitesimal had been found to be incoherent and was 'banished from mathematics' through the work of Weierstrass and others" (2006, 2), views which, for Ehrlich, are historically inaccurate, given the existence, from the early 1870s on of "a large, diverse, technically deep and philosophically pregnant body of consistent non–Archimedean mathematics of the (non–Cantorian) infinitely large and the infinitely small" (ibid., 3). Somewhat similarly Bell discusses Part V of PoM as "a kind of paean to Weierstrass and Cantor" (2005, 168), suggests that Russell ignored the work of mathematicians who were developing notions of infinitesimals (ibid., 176), and indicates that Russell would be "greatly surprised—perhaps even dismayed" at the resurgence of mathematically respectable notions of infinitesimals in recent years (ibid., 184).

[27] Similarly, in "Recent Italian Work on the Foundations of Mathematics", begun, apparently, in October 1900, Russell writes that the "school of Italian philosophical mathematicians" use "a rigid formalism which leaves no opening for the pernicious influence of obviousness", and then adds:

> Those who know how many obvious mathematical propositions are false and how many highly paradoxical propositions are true, and how difficult it is, in verbal reasoning, to avoid unconscious employment of an obvious proposition, will appreciate the reasons for banishing all words from our deductions, and effecting everything in a wholly symbolic language. (1901f, 352)

For Russell's use of "paradoxical" here, see below.

[28] Russell describes a somewhat similar process in geometry, where attempts to prove the "axiom of parallels" led to the development of non–Euclidean geometries (see PoM, 373).

[29] Russell notes, however, that in order to prove that "the similarity of part and whole" is "impossible" for finite classes, a different definition of the distinction between infinite and finite class should be used "to avoid tautology" (PoM, 360, first footnote).

[30] See, for example, the passage cited in note 27 above and other examples below.

[31] For similar discussions of arriving at the axioms of logic, see PM, Vol. I, 59–60 ("Reasons for Accepting the Axiom of Reducibility") and 1924a, 165–6.

[32] See, for example, Irvine (1989 and 1994) and Hylton (1990, 321–5).

[33] In his 1999b, Schwerin, discusses a tension in PoP, between a conception of philosophy as enabling us to attain certainty via attention to immediate knowledge gained through acquaintance, and the view Russell presents in the final chapter according to

which the "value" of philosophy "is to be sought largely in its very uncertainty" (PoP, 156) and in its power to challenge our preconceptions. Schwerin argues that "this tension is due in large measure to Lady Ottoline's influence, and to [Russell's] wish to reconcile his thought to hers" (1999b, 28). However, as I have presented Russell's development, whatever personal motivations may have been involved in the composition of PoP, the tension Schwerin points to reflects that between Russell's Moorean and post–Peano philosophies.

[34] Hylton, who has only a brief discussion of Russell's post–Peano acceptance of Cantor's theory of the transfinite (1990, 192–5), observes, as one of the two points he emphasizes there, how accepting the views of Cantor and Weierstrass supports Russell's Moorean metaphysics (ibid., 194–5), thereby supporting his general view that what Russell gained from the Paris Congress "enable[d] him to defend his [Moorean] doctrines" and to "show that they could play a role in the solution of problems which had previously seemed insoluble" (ibid., 153). While Russell does thus use the views of Cantor and Weierstrass to support his Moorean metaphysics, I am arguing here that his acceptance of their views undermines his Moorean epistemology along with his Moorean conception of the proper "method" in philosophy and of the relation between philosophy and science.

[35] This had been Russell's own pre–Peano practice (see, for example, 1899–1900, 72).

[36] Russell actually leaves it as an open question as to whether space exhibits this structure (see PoM, 444; also OKEW Lecture V, especially 147–50).

[37] As Quine suggests, this sort of view of justification leads to the view that for philosophers to appeal to science, in particular, psychology, when doing epistemology would involve "circular reasoning" (see 1969, 75–6).

[38] In all the passages I cite below, Russell presents Cantor as having definitively resolved all the traditional problems of infinity and continuity; however, in a recently discovered manuscript, apparently written for *A History of Western Philosophy*, Russell indicates that set–theoretic paradoxes call into question the status of Cantor's achievement. In particular, he writes that resolving the paradoxes requires adopting "methods which throw doubt on things which have been accepted in mathematics since the seventeenth century", so that "the solution of a problem, it should seem, consists only in reducing it to another problem more difficult than the first" (Vianelli, 10).

[39] They are also compatible with the view Russell expresses at the end of PoP that "as soon as definite knowledge concerning any subject becomes possible, this subject ceases to be called philosophy, and becomes a separate science", so that "those questions which are already capable of definite answers are placed in the sciences, while those only to which, at present, no definite answer can be given, remain to form the residue which is called philosophy" (PoP, 155), and likewise at the end of PLA that "the only difference between science and philosophy is, that science is what you more or less know and philosophy is what you do not know. Philosophy is that part of science which at present people choose to have opinions about, but which they have no knowledge about" (PLA, 243). Insofar as Russell takes Cantor to have definitively resolved the problems of infinity and continuity, he would thus seem to regard him as having removed these problems from philosophy to science. In his 1950 essay "Logical Positivism", after discussing the work of Carnap and Tarski, Russell writes that the "vast technical development of logic, logical syntax, and semantics . . . has become so technical, and so capable of quasi–mathematical definiteness, that it can hardly be regarded as belonging to philosophy as formerly understood" and that "on the same ground" that Newton's "natural philosophy" is no longer regarded as part of philosophy "even though it solves problems that *were* philosophical

problems", "much of recent work on logic, syntax, and semantics should be regarded as definite knowledge, not philosophical speculation" (1950, 159).

[40] Even though he hopes, as I have discussed above, that freed of past theory, we will, in fact, agree as to what we "intuit".

[41] This is perhaps because, given the occasion of that lecture, he is concerned to emphasize the ways in which he holds that certain philosophers, Spencer among them, have misappropriated the results of science (see especially 1914b, 60–4).

[42] In identifying being "equal in number" with "similarity" defined in terms of one–to–one correspondence, Russell reflects his post–Peano acceptance of Cantor's theory of the transfinite. For, as I have discussed above, he formerly denied, for example, that the class of whole numbers and the class of even numbers have the same cardinal number or are "equal in number", even though the members of those classes may be placed in a one–to–one correspondence with each other; for he denied that those classes have any number at all.

[43] As a number of commentators have noted, there is a tension in Russell between a sort of "structural" logicism he advances in some places (especially when he presents geometry as part of logic) and the sort of logicism that depends on defining, for example, the cardinal numbers in logical terms. For some discussion of this topic and how these different views of logicism are related to the composition of PoM, see Byrd (1999, 44–54).

[44] "More" perspicuously, because a fully perspicuous representation of that proposition would have to take into account the definition of "similarity" in terms of one–to–one correspondence.

[45] Which will be reflexive over the relevant entities in question.

[46] The technical difficulty in defining moments and points is that since the "events" in terms of which they are to be defined "have a finite extent", events can be "overlapping" without being entirely simultaneous, so that moments (and points) will have to be "constructed" out of "overlapping", rather than "simultaneous" event particles. Russell credits Whitehead with the solution of this problem (see, for example, OKEW, 114ff; 1924a, 166).

[47] In my (2007), I compare how Russell and Frege understand relations among sentences of the forms (Ab_1), (Ab_2), and (Ab_3); however, I do not there discuss issues I focus on below regarding how Russell's post–logicist understanding such sentences bears on his Moorean conception of analysis.

[48] In PM, Russell had dispensed with classes by accepting the "no classes" theory; however, in the passage I have quoted from OKEW, Russell makes no allusion to his dispensing with classes and instead reflects the understanding of the "principle of abstraction" he had at the time of PoM. Thus, as early as December 1903, Russell wrote to Couturat that once he proves his earlier "axiom of abstraction" by substituting an equivalence class of objects for the "hypothical quality common to all these objects" ("*substítuer la classe même des objects dont il est question á la qualité hypothétique commune á tous ces objects*"), it would be better to call the "principle of abstraction" the "principle replacing abstraction" ("*princípe remplaçant l'abstraction*"). See Schmid (2001, 346–7).

[49] Thus, if $E(\alpha,\beta)$, where E is transitive and symmetric, then for any x, if $E(x,\alpha)$, then (since, by assumption $E(\alpha,\beta)$ and E is transitive) $E(x,\beta)$; and given $E(\alpha,\beta)$, then (since E is symmetric), $E(\beta,\alpha)$, in which case for any x, if $E(x,\beta)$, then (since E is transitive together with $E(x,\beta)$ and $E(\beta,\alpha)$), then $E(x,\alpha)$. Thus, if $E(\alpha,\beta)$, then, $(\forall x)(E(x,\alpha) \longleftrightarrow E(x,\beta))$, in which case—by naïve set theory—$\{x : E(x,\alpha)\} = \{x : E(x,\beta)\}$. In PoM (305), Russell indicates that "the proof of the principle of abstraction ... is philosophically subject to the doubt resulting from the contradiction [that is, Russell's paradox] set forth in Part I, Chapter X."

[50] See note 21 above.

[51] In his 1884, §69, Frege acknowledges that we do not typically think of numbers as classes, or "extensions of concepts"; and in that case, we would not typically ascribe to numbers properties—such as having elements or as having more elements than a given class—that we typically assign to classes. As G. H. Moore notes (CP3, xxvii), insofar as Russell was led to regard cardinal numbers as classes of similar classes as a result of considering the view that Peano rejects in the paper Russell cites in this passage, then Russell was probably indirectly influenced by Frege's view, since Peano was probably led to discuss the view he there criticizes by his knowledge of Frege's view, having reviewed Frege's *Grundgesetze* in 1995.

[52] As I have mentioned above (note 6), during his Moorean period, Russell regarded cardinal numbers as properties of plural subjects.

[53] While Whitehead and Russell do not consider alternative set–theoretic definitions of the sort introduced by Zermelo and von Neumann, it seems clear that they would have no difficulty in acknowledging that such definitions are possible and would take the possibility of such definitions as confirming their view that the ordinary use of numerical expressions is "vague", so that it is up to us to choose which "precise" meaning to assign to such terms, much in the same way that Quine uses the possibility of alternative set–theoretical definitions of numerical expressions to support his thesis of the "inscrutability of reference". (See in this connection note 67 below.) Note that in addition to mentioning alternative set–theoretic definitions of numbers, Quine also mentions (1969, 43f) the possibility of assuming "the natural numbers themselves" without defining them in set–theoretic terms at all; this amounts to treating them as indefinables of the sort that Russell originally took numbers to be and that he comes to regard as the main alternative to his definition of numbers as classes of similar classes. What is needed to establish the "vagueness" (in Russell's sense) of numerical terms, or the "inscrutability" of their reference (in Quine's terms) is simply that there are different ways to assign reference to such terms compatible with the mathematical statements we wish to affirm, not that those different assignments have to all involve set–theoretic definitions.

[54] More generally, in OKEW, 124–6, Russell provides this sort of defense of all definitions of the form ($\mathrm{Ab}_{\mathrm{df}}$),

[55] Later in that typescript (25), Moore indicates that he regards Russell's theory of "On Denoting" as presenting a better account of what is "before our minds" when we apprehend the propositions analyzed there than the account he had previously accepted in PoM.

[56] Accordingly, it perhaps is no surprise that in his unpublished review of PoM, Moore writes that neither of the two definitions of infinity (the first, derived from Dedekind, that an infinite number is the number of a class which is similar to a proper subset of itself, the second derived from Cantor, according to which an infinite number is a number which cannot be reached by starting with 0 and successively adding 1) used by Russell reflect "the property" (namely, "endlessness") which the word "infinity" "most naturally suggests" and which is "undoubtedly the most easily intelligible of any which the word can suggest" (1905–6?, 29); and he attempts to arrive at an "equivalent" definition of infinity (according to which "a series is infinite, if and only if it *either* is itself endless *or*

contains an endless series as a part of itself") which "is far more in accordance with the ordinary use of the word 'infinity'"(ibid., 30–1) than the definitions used by Russell.

[57] See, for example, 1919c, 139–42; 1921, 180–4, 220–3; 1927a, 1–3;1927b, 220–23.

[58] Thus, for example, Williamson (1994, 52) traces Russell's concern with vagueness only to as far back as 1913, while Faulkner writes: "[V]agueness had hardly figured as philosophically important in [Russell's] earlier writings. In *The Principles of Mathematics* (1903) and *Principia Mathematica* (1910–13) there is next to nothing on vagueness." (2003, 43) While Schwerin notes the use of the term "vague" in PM (Volume I, 12), he argues that since that is the only use of that term in PM and occurs "only in passing . . . students looking for insights into Russell's *Principia* conception of vagueness will be better served looking elsewhere" (1999a, 52–3).

[59] See here Hyde's gloss on Russell's characterization of vagueness: "A representation if vague just if there are, or could be, various different referents compatible with the representation given. On the basis of the representation itself one cannot determine its referent—various possibilities remain open, the representation lacking the distinctness which would entail any particular referent as being represented." (1992, 147)

[60] The charge is often made that Russell fails to distinguish between vagueness and generality (see, for example, Black (1937, 432, note 22), Kohl (1969, 37–8), Rolf (1982, 71)). However as Hyde (1992, see, for example, 148–50 and 153–7) makes clear, once it is recognized that Russell takes a predicate as purporting to designate a property rather than the individuals to which it applies, then what determines, for Russell, whether a predicate is precise is whether is it succeeds in designating a unique property, so that a predicate can be general and precise if it designates a unique property which in turn applies to more than one individual, and whether it is vague is determined not by how many individuals it applies to (or can apply to) but how many properties it may be interpreted as designating.

[61] Before he adopted his "no classes" theory, enabling him to deny that numerical expressions are referring expressions altogether.

[62] Or Quine's.

[63] In his 2007 (102–3) Hylton presents Russell's "supreme maxim of scientific philosophizing" as a case in which Russell is applying his "Principle of Acquaintance". Moreover, he presents Russell as holding that in all cases of analysis, "the fully analysed sentence corresponds to the thought which is expressed by the ordinary, unanalysed sentence", in particular that "the fully analysed sentence has a structural correspondence with something which is psychologically real" (93). And further he finds it puzzling for Russell to be agnostic regarding the existence of such entities as classes (see 105, note 1). On all these points, I believe that Hylton wrongly assimilates Russell's pre–Peano Moorean conception of analysis to his post–Peano style of "substituting logical constructions for inferred entities" (although, as I discuss in note 65 below, I believe that Russell's theory of descriptions, which Hylton takes as the exemplar of Russellian analysis, differs in a number of respects from his post–Peano "logical constructions"). Accordingly, I believe also that Linsky (2007, 109, 114–5) and similarly Sainsbury (1979, 298–301) are right to argue that some of Russell's post–Peano "logical constructions" should not be construed as providing accounts as to what our ordinary sentences in the relevant domain mean, or what we are acquainted with when we understand those ordinary sentences. However, given that Russell's "logical constructions" cannot be made coherent so long as he remains committed to such principles as (Aug) and (PoA), then while Linsky and Sainsbury rightly argue

that many of Russell's characteristic post–Peano "analyses" do not reflect the Moorean conception of analysis, Hylton is right to suggest that there are difficulties in reconciling those post–Peano "analyses" with Russell's "official" (pre–1919) view of analyses. Note also in this connection that in a 1944 reply to Max Black, Russell writes:

> I come now to the question of logical constructions. Mr. Black connects this much more closely than I should do with my doctrine that sentences we can understand must be composed of words with whose meaning we are acquainted. My first applications of the method of logical construction were in pure mathematics: the definitions of cardinals, ordinals, and real numbers, and the construction of points in a projective space as pencils in a descriptive space. All these antedated the theory of descriptions, and were dictated by dislike of postulation where it can be avoided. This motive remains, quite independently of my later introduction of acquaintance. (1944b, 692)

Here, I believe that while Russell correctly indicates that his "logical constructions" should not be understood as applications of (PoA), it is misleading for him to claim that these "constructions" precede his "introduction of acquaintance". For, as I have argued, Russell introduces the notion, if not the term, "acquaintance", during his Moorean period, before his post–Peano "logical constructions". More generally, it is misleading for him to suggest that his "logical constructions" are independent of, and thus consistent with, (PoA); instead, insofar as he accepts (PoA), he should not carry out his "logical constructions", and insofar as "analysis" should follow the model of his "logical constructions", he should reject (PoA).

[64] See also TK (6), where Russell relates his analysis of "experience" to the "vagueness" of that term and OKEW (211), where he characterizes "philosophic analysis" in general terms as beginning with "data" which are "vague".

[65] Russell's theory of descriptions differs in a number of ways from his post–Peano "logical constructions". First, of all, Russell often presents that theory in conjunction with emphasizing (PoA); see, for example, OD, 415, 427; 1911b and note 10 above. Moreover, Russell suggests (1911b, 155) that his analysis is meant to reflect "what is actually in my mind" when I understand a statement involving a definite description or ordinary proper name, such as "Julius Caesar". Further, in PM (Vol. I, 72) Russell contrasts "the case of descriptions", in which "it was possible to *prove* that they are incomplete symbols" with "the case of classes" in which "we do not know of any equally definite proof" and in which "it is not necessary . . . for our purposes to assert dogmatically that there are no such things as classes". Also, in OD, Russell takes himself to present decisive arguments against other proposed analyses of the propositions in question. However, as against this, Russell acknowledges in OD that his "interpretation" of propositions expressed by sentences of the form "The *F* is *G*" "may seem . . . somewhat incredible" (417), and in his 1957 reply to Strawson, he writes:

> This brings me to a fundamental divergence between myself and many philosophers with whom Mr Strwson appears to be in general agreement. They are persuaded that common speech is good enough, not only for daily life, but also for philosophy. I, on the contrary, am persuaded that common speech is full of vagueness and inaccuracy, and that any attempt to be precise and accurate requires modification of common speech both

> as regards vocabulary and as regards syntax. . . . My theory of descriptions was never intended as an analysis of the state of mind of those who utter sentences containing descriptions. . . . I was concerned to find a more accurate and analysed thought to replace the somewhat confused thoughts which most people at most times have in their heads, (see 1959, 241–243)

Thus, while there are some reasons for holding that Russell originally intended for his theory of descriptions to conform to the Moorean conception of analysis, however he originally regarded that theory, he eventually comes to regard it as conforming to his other post–Peano analyses in which the goal is to "pass from the vague to the precise". For some further discussion of the issues raised here, see Szabó (1905, section 2) and Kripke (2005, 1107, note 208).

[66] Hylton concludes his 1996 (47–8), by quoting this passage from Quine and arguing that it rejects presuppositions regarding propositions and their structure that were taken for granted by Russell. While I agree that this passage rejects presuppositions of Russell's early Moorean conception of analysis, my claim is that it is in accord with Russell's post–Peano practice of analysis and also with his post–1918 views of meaning and understanding which make that practice of analysis coherent.

[67] In his "The Many Lives of Ebenezer Wilkes Smith", Vann McGee presents Russell's 1923a account of the vagueness of the name "Ebenezer Wilkes Smith" as a case of "the inscrutability of reference" consistent with "the line of argument of argument of Chapter Two of *Word and Object*" (2004, 621). I have argued, in effect, that in his earlier account of the vagueness of numerical terms—specifically, in his acknowledging that if there are indefinables of the sort he previously took cardinal numbers to be, then either those indefinables, or classes of similar classes will enable us to sustain the same "formulae of arithmetic"—Russell is likewise arguing for a case of the "inscrutability of reference".

[68] Here, I recognize, I have considered only a few aspects of Russell's post–1918 views and how they are related to his earlier philosophy. Stevens (2005, Chapters 5; see also 2006b) discusses another aspect—namely, how Russell's engagement with Wittgenstein contributes to his willingness to "psychologize" propositional contents in his post–1918 writings. I recognize also that in addition to the similarities between Russell's post–1918 views and Quine's that I have pointed to, there are also significant differences. One such difference is Russell's continuing adherence (from 1910 on) to a correspondence theory of truth and the notion of "fact", which contrasts with Quine's acceptance of a "deflationary" view of truth.

[69] Thanks to the participants of the Riga conference and an anonymous referee for helpful comments.

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