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INTRODUCTION

This paper is composed of two independent parts. The first is concerned with Russell's early philosophy of mathematics and his quarrel with Poincaré about the nature of their opposition. I argue that the main divergence between the two philosophers was about the nature of definitions.

In the second part, I briefly present Leśniewski's Ontology and suggest that Leśniewski's original treatment of definitions in the foundations of mathematics is the natural solution to the problem that divided Russell and Poincaré.

1. RUSSELL AND POINCARÉ

1.1. Russell

In 1903, Bertrand Russell published a peculiar work, *The Principles of Mathematics*, which history will recall as one of the founding texts of the philosophical movement that would later be called *logicism*. The *Principles* is peculiar for a number of reasons. The most well known is that it provides the first clear presentation and analysis of the paradox of the set of all sets that are not members of themselves. Russell encountered this paradox as the text was almost completed.

His presentation and analysis are the subject of the tenth chapter - which is only five pages long - and of the second appendix where Russell presents a sketch of a solution. He seemed not to realize the scope of this discovery, for the work does not display any sense that its project is being confronted with a "Foundational Crisis". On the contrary, the work optimistically maintains the following thesis: all mathematics and part of physics are reducible to a few logical principles or, in other words, are logical constructions, which ultimately rest on primitives amongst which we define relations.

The discovery of the paradox, however, and its apparent insolvability were potentially catastrophic for Russell's project: the reduction of mathematics to logic. This reduction meant for Russell, during this period, that

> Pure Mathematics is the class of all propositions of the form "p implies q" where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants.[1]

Hence, Russell found himself in a delicate position. His work was supposed to convince the scientific community that a rigorous analysis allows us to ground the entirety of mathematics on a very small number of purely logical principles. But he then shows that this small number of foundational principles is incoherent. As Poincaré would later say "Logic is no longer sterile, it fathers antinomy."[2]

Another peculiarity of the work is the scope of the task that Russell aims to accomplish: the task of reduction or reconstruction, depending on which view we adopt, is far reaching.

Starting from irrefutable logical principles, the first part of the text is not a reconstruction at all, but a regressive analysis that brings its author to a logical grammar (the fourth chapter bears the title *Proper names, adjectives and verbs*), and to the detailed examination of the ideas of class, propositional function, variable, relation, and finally, contradiction. Russell's philosophical position in this work could be called naïve hyperrealism, as his ontology abounds with all sorts of beings. Everything *is* or exists, from humans to numbers, including logical constants. The fundamental concept is that of the *term*.

> Whatever may be an object of thought, or may occur in any true or false proposition, or may be counted as *one*, I call a *term*. This, then, is the widest word in the philosophical vocabulary. I shall use as synonymous with it the words unit, individual and entity. The first two emphasize the fact that every term is *one*, while the third is derived from the fact that every term has being, *i.e. is* in some sense. A man, a number, a class, a relation, a chimaera, or anything else that can be mentioned, is sure to be a term; and to deny that such and such a thing is a term must always be false.[3]

What is important to take note of here is the introduction of the concept of *Being*. At this period, Russell defends a realism that could be fairly characterized as unbridled.[4] In the realm of being, as understood by Russell, it is possible to find anything one is looking for, probably even square circles, at least until 1905 with the publication of "On Denoting".[5] This hyperrealism, which Russell would later repudiate in the second edition of the *Principles*, plays a very central role in his early philosophy of mathematics:

> At the time when I wrote the "Principles", I shared with Frege a belief in the Platonic reality of numbers, which, in my imagination, peopled the timeless realm of Being. It was a comforting faith, which I later abandoned with regret.[6]

He also wrote in *The Principles:*

> Hence Adam's first thought must have been concerned with the number 1; for not a single thought could precede this thought. In short, all knowledge must be recognition, on pain of being mere delusion; Arithmetic must be discovered in just the same sense in which Columbus discovered the West Indies, and we no more create numbers then he created the Indians.[7]

Russell's realism must, therefore, be taken very seriously as it is a presupposition that grounds the entire work. This philosophy no

doubt had an effect, not only on his ontology, but also on his epistemology and, finally, on the question that concerns us, his conception of the nature of definitions.

It should be highlighted that for Russell definitions cannot be creative, in the sense that a definition cannot have the consequence of bringing a new object into the world of being, consequently enriching the ontology. For him any definition is *nominal*, in a way that will be specified.

There are three important passages in the *Principles* in which the question of the nature of definitions is taken up directly. The most important of these passages is one in which Russell defines cardinal numbers and criticizes the definition from abstraction put forward by Peano.

The foundational ideas that he relies upon are those of (1) a *class concept*, which is a predicate considered from the point of view of its determining a class; and (2) of a *one-one* relation between classes, i.e. a bijective function. Each class concept defines a class, so according to Russell, "Numbers, then, are to be regarded as properties of classes".[8] He first defines what it means for two classes to have the same number.

> Two classes have the same number when, and only when, there is a one-one relation whose domain includes the one class, and which is such that the class of the terms of the one class is identical with the other class.(...) When two classes have the same number, they are said to be *similar*.[9]

This definition, a variation on Frege's, is slightly more complex, the aim being to also take into account empty classes. After noting that similarity is a reflexive, symmetrical and transitive relation, he introduces the Peanian definition of number itself:

> Now these three properties of a relation are held by Peano and common sense to indicate that when the relation holds between two terms, these two terms have a certain common property, and *vice versa*. This common property we call their number. This is the definition of number by abstraction.[10]

Thus, for example, all classes having exactly three members have a common property, that of having three members. This common property *is* the number 3. Russell believed that this definition of number from abstraction suffered from a fatal flaw, that of not ultimately determining the number of a class. In effect, no matter what common property all similar classes have that is unique to these similar classes, according to the definition from abstraction this is the number of this class. Consider, for example, the following relation: All classes having three members and only them are also related to the computer I am using now. This relation surely has being just as any other relation and it satisfies the definition of the number 3 by abstraction. But then, any object whatever can be the number 3. The definition from abstraction determines a class of terms that all have the common attribute of being related to similar classes, as being related to these classes andno others. There are two possible ways of remedying this problem. The first consists in considering the number of a class to be a class of all the entities that have this property of being related to all the classes that are similar and only those.

> But this method is practically useless, since all entities, without exception, belong to every such class, so that every class will have as its number the class of all entities of every sort and description.[11]

The solution that Russell puts forward is one that he believes can be applied universally anywhere that the definition from abstraction is applicable, it is as follows:

> This method is, to define as the number of a class the class of all classes similar to the given class. Membership of this class of classes (considered as a predicate) is a common property of all the similar classes and of no others; moreover every class of the set of similar classes has to the set a relation which it has to nothing else, and which every class has to its own set.[12]

This, according to Russell, solves the problem. Russell himself was shocked to learn that Peano had thought of adopting this definition

and finally decided to reject it. It seems that for Peano, intuitively speaking, the class of similar classes to any given class has properties that the number that we are seeking to define does not have, although Russell admits that he does not know which ones. Russell insists that the number is itself a class of classes and not a class concept which determines this class of classes. This is because one class of classes corresponds to more than one class concept. For example, the class of classes similar to the class of the stars in our solar system is the same as the class of classes similar to the class of the heads of the Roman Catholic Church at a given time (since the Sun is the only star in our solar system and there is only one Pope at any time in history), but their class concepts are totally different.

This stance on the nature of numbers is fundamental for Russell's philosophy of mathematics. At the very end of the work (apart from the appendices), actually in the conclusion, Russell sums up the entirety of the work and makes the following statement, which is worth repeating:

> A definition is always either the definition of a class, or the definition of the single member of a unit class: this is a necessary result of the plain fact that a definition can only be effected by assigning a property of the object or objects to be defined, *i.e.* by stating a propositional function which they are to satisfy. [. . .] And wherever the principle of abstraction is employed, *i.e.* where the object to be defined is obtained from a transitive symmetrical relation, some class of classes will always be the object required.[13]

For Russell this idea of definition corrects a fundamental problem with the definition by abstraction, which remains definitively ambiguous. We can conclude that, at the time of the *Principles*, definitions are not, for Russell, creative in the sense that a definition does not generate a new term. The reason rests on Russell's realism. For Peano from the class of all similar classes we can abstract a new object, the number of members of these similar classes. But for Russell, given any object, *there is* a relation (not we can define a relation, it is already there) such that all and any of the similar classes have the relation to that object and to no other. Terms for Russell are eternal and immutable. Must we therefore conclude that definitions are sterile? Russell recognizes the problem that this could engender. Insofar as his ultimate goal is to show how all of mathematics can be reconstructed out of a small number of logical primitives by use of ever more complex definitions, he cannot accept that definitions are sterile, or if they are it is in a trivial way: a definition is not a construction, it is an identification. Curiously, Russell takes up this question at the beginning of the work, in the chapter entitled *Denoting* (which has nothing to do with the concept of denotation which appears in "On Denoting").[14]

> It is a curious paradox, puzzling to the symbolic mind, that definitions, theoretically, are nothing but statements of symbolic abbreviations, irrelevant to the reasoning and inserted only for practical convenience, while yet, in the development of a subject, they always require a very large amount of thought, and often embody some of the greatest achievements of analysis. This fact seems to be explained by the theory of denoting. An object may be present to the mind, without our knowing any concept of which the said object is *the* instance; and the discovery of such a concept is not a mere improvement in notation. The reason why this appears to be the case is that, as soon as the definition is found, it becomes wholly unnecessary to the reasoning to remember the actual object defined, since only concepts are relevant to our deductions. In the moment of discovery, the definition is seen to be *true*, because the object to be defined was already in our thoughts; but as part of our reasoning it is not true, but merely symbolic, since what the reasoning requires is not that it should deal with *that* object, but merely that it should deal with the object denoted by the definition.[15]

It is unlikely that Russell's theory is entirely coherent. However, it successfully accomplishes the task of reduction and implies a minimal epistemology, that of recognition of what is already there. Later, the famous contradiction will complicate matters, but for the time being Russell, strangely, did not recognize this.

1.2. Poincaré (and Russell)

Two years later, while he was still contemplating the famous contradiction, Russell published a review of the English translation of Poincaré's work *La science et l'hypothèse* in *Mind*. This review unleashed a controversy in which, though not explicitly the center of attention, the question of the status of definitions and of whether they are sterile or useful, would be extensively debated. It is necessary to say that the tone adopted at once by Russell is that of somebody who looks for quarrel. This is obvious from the beginning of the first paragraph.

> In this book, which consists in the main of previous articles somewhat re-written, M. Poincaré's well-known merits appear to the full – his lucid and trenchant brevity, his air of easy mastery, which often makes his thought appear less profound than it is, and his power of co-ordinating the whole domain of mathematics and physics in a single system of ideas. But these merits, great as they are, are accompanied by what cannot but appear as defects to any one accustomed to philosophy.[16]

The tone that is adopted announces from the beginning that the ensuing debate will be fruitless. Even though Russell and Poincaré use the same vocabulary they are not really talking about the same thing.

The first edition of *La science et l'hypothèse* dates from 1902 and the English translation from 1905. The first chapter is entitled "Sur la nature du raisonnement mathématique". Poincaré defends the following thesis: logic, because of the tautological nature of syllogistic reasoning, is sterile.

> Nothing essentially new can be learned from syllogism and, if everything results from the principle of identity, everything should be able to come down to it. Will one admit that all these theorems that fill so many volumes be only diverted manners to say that A is A?[17]

One thing that should be pointed out is that when Poincaré talks about logic he is talking about Aristotelian logic as it had been handed down to us and transformed throughout history. He is not talking about the new logic, or as it was called at the time "logistic", which started with Boole, and that Frege and Russell would eventually render important through their propositional calculus. In fact, he seems to make no distinction between ancient and modern logicians.

A second thesis that Poincaré puts forward is that mathematical science, in opposition to logic, is creative, and that its creative power lies essentially in inductive proof (Poincaré uses the now outmoded expression "raisonnement par récurrence").

> The essential character of mathematical induction is that it contains, condensed so to speak in a unique formula, an infinity of syllogisms. (...) We thus see that, in the reasoning by induction, we just express the minor premise of the first syllogism, and the general formula which contains as particular cases all the major premises.[18]

The validity of inductive reasoning cannot be proved. It comes from the power of the mind which is capable of conceiving an indefinite repetition of similar acts.

> This rule, inaccessible by analytical demonstration and by experiment, is the genuine type of the a priori synthetic judgment. On the other hand, we could not think of it as a convention as for some of the postulates of geometry.[19]

Despite their exchanges, Russell and Poincaré will never arrive at a point of agreement, for the gulf that separates their ways of thinking is too great. On Poincaré's side, induction is based on the irreducible synthetic a priori intuition that some act of the mind can be repeated indefinitely and this is equivalent to an infinity of syllogisms. For Russell, there is no problem here, because there is no problem with endless processes.

> The property of the mind which is in question is, therefore, this "It is possible to add 1 to any number whatever". But this does not yield us the principle of mathematical induction. Which says not merely that the addition of 1 will always give a number, but that *every* natural number can be obtained from such addition starting from 0. It limits the natural number at the same time that it shows the series of them to be endless: they all appear in this series, any point of which can be reached by successive steps starting from 0. Now this limitation, which is what is really used when proofs are conducted by mean of mathematical induction, is not a synthetic *a priori* intuition, or a property of the mind, or a condensation of an infinite number of syllogisms, it is merely the *definition* of a finite number.[20]

The question of the nature of definitions would however gravitate towards slightly different contexts; that of the unpredictability of certain definitions, those of properties that do not determine classes. In fact, in 1905, Russell presented an article in which he proposed three possible solutions for eliminating these paradoxes.[21] In this article Russell presented sketches of three promising theories for avoiding contradiction. These are (a) the zigzag theory, (b) the theory of limitation of size and (c) the no class theory. One interesting thing that Russell presents is a matrix, which he thinks applies to all cases of contradiction based on unpredictability.

> Given a property ϕ and a function f, such that, if ϕ belongs to all the members of u, $f`u$ always exists, has the property $\phi$, and is not a member of $u$; then the supposition that there is a class $w$ of all terms having the property $\phi$ and that $f`w$ exists leads to the conclusion that $f`w$ both has and has not the property ϕ.[22]

Here is a classical example using contemporary notation. Let the barber in some town be the man who shaves all and only those men that do not shave themselves. Let $\phi(x)$ stand for x shaves x. If u is such that if $(\forall x)(x \in u \supset \phi(x))$ then $\phi(u)$ exist (it is the barber), $\phi(f(u))$ (the barber shaves itself) and $f(u) \notin u$ (the barber is not among the selected class u). Up to this point, there is no contradiction because the barber may come from another town. Now suppose that we restrict ourselves to all the men of a given town. In that case $w = \{x : \phi(x)\}$ and $f(w)$ exists (there is a man in town who is a barber i.e. a man who shaves all men and only those men that do not shave themselves). This leads to a contradiction: Suppose that $\phi(f(w))$. Then $f(w) \notin w$ and because $w = \{x : \phi(x)\}$, $\neg\phi(f(w))$. But $(\forall x)(x \in w \supset \phi(x))$, thus $f(w)$ exists and $\phi(f(w))$. In such a case the solution is quite simple: there is no barber as defined above. The situation is more complex if we start with x is a member of x instead of x shaves x. But nevertheless, some sets are like the barber, they do not exist.

The task for Russell is to find a natural and intuitive way of blocking the formation of these kinds of monsters. This is what the three approaches should be able to do. The zigzag theory consists in axiomitazing the idea of a predicative function. This approach was abandoned rapidly as it lacks directing principles, apart from the one that avoids contradiction.[23] The second theory uses also the idea of predictability but it is no longer the complexity of the property that will be the source of its unpredictability, but rather the size of the class to be defined. This approach was suggested to Russell by the Burali-Forti paradox. Here again he will not make much progress. Only the third approach will be retained by Russell. At the moment of publication he would add the following note:

> [*Note added 5th February 1906*. From further investigation I now feel hardly any doubt that the no-classes theory affords the complete solution of all the difficulties stated in the first section of this paper.][24]

In this embryonic theory, which would later become the type theory, Russell no longer considers propositional functions as primitive and chooses instead the idea of a proposition, leaving propositional functions to be reconstructed. Let p be a proposition and a one of its constituents. '$p(x/a)$' denotes what p would become if each occurrence of a were substituted for x at each occurrence within p. We can use this notation to maintain more general quantified statements like '$p(x/a)$ is true for every value of x'. This reconstructed propositional function is independent of a in the following manner. Let $q = p(b/a)$. It is therefore equivalent to state '$p(x/a)$ is true for every value of x' and '$q(x/b)$ is true for every value of x'. It is obvious that '$p(x/a)$' and '$p(x/b)$' have the same structure. However, the more interesting consequence of this finding is as follows:

> Here the values of x for which $p(x/a)$ is true replace the class u; but we do not assume that these values collectively form a single entity which is the class composed of them.[25]

We are far from the ontology of the *Principles*. Russell's philosophy of definitions seems to have taken a radical turn. Do not forget that this article was written two or three months after the publication of "On Denoting". We may first notice a sort of "linguistic turn". Definitions of the sort $p(x/a)$ are no longer mere truisms. The syntax of language and even grammar take a dominant role. Contradictions can be avoided because we no longer postulate that an object u, which would be the class of values that makes $p(x/a)$ true, preexists in some realm of Being. There are combinations of symbols which seem to make sense but fail to denote anything. Definitions take on a new status, even if this status is not made explicit.

Poincaré vehemently attacks Russell's theories in an article published in *La Revue de métaphysique et de morale* entitled "Les mathématiques et la logique". It should be noted that Russell is a bit inconsistent: maintaining three theories in one text, with each of the theories being inconsistent with the next, is an exploit that few philosophers have been able to accomplish.

What is most interesting about Poincaré's article is not its attack of Russell's proposals but his solution to the problem of paradoxes. A definition should not contain *vicious circles*, meaning that the definition of a set of objects E should not make reference to the set E as though it predated its definition. This implies, as we have already seen, that for Poincaré definitions are not trivial, they create objects, on the condition that we respect certain principles such as the non-circularity principle.

Poincaré goes even further. For him even the definition of the inductive number, i.e. of natural numbers by induction is not predicative.[26] For him, the only way to avoid contradictions is to respect the principle of non-circularity in formulating definitions.

It is interesting to examine Russell's response in "Les paradoxes de la logique".

> M. Poincaré holds that theses paradoxes all spring from some kind of vicious circle, and in this I agree with him.

> But he fails to realize the difficulty of avoiding a vicious circle of this sort. I shall try to show that, if it is to be avoided, something like my 'no-class' theory seems necessary; indeed, it was for this purpose that I invented the theory.[27]

Thus, Poincaré failed to notice the difficulty involved in avoiding the vicious circle simply by prohibiting the use of circular definitions. According to Russell, this difficulty is as follows:

> The vicious circle he [Poincaré] proposes to avoid by defining E as 'all numbers definable in a finite number of words without mentioning E.' to the noninitiated, this definition looks more circular than ever.[28]

The problem for Russell is that he must characterize the constraints that must be imposed on propositional functions in order that they give rise to classes in an independent way. It is not sufficient to say that the vicious circle must be avoided. The fundamental logical principles, once respected, must ensure that we are not dragged into the vicious circle. But this should be a consequence of the soundness of the logical principles, not a principle in itself. Russell's fundamental idea is that each proposition related to a class can be paraphrased by a proposition related to the value of variables that satisfy a propositional function. What he used to consider the *definition* of a class he now views as defining the *range* of a propositional function, but the latter is not an *object* suitable to be taken as a value for the propositional function. In fact, Russell would later arrive at a certain reification of classes, but these cannot be values of the propositional function that engendered the classes, only propositional functions of a superior type. This will be the *Type Theory*. For the moment, what is important is that a reform of syntax is necessary in order for the possibility of contradiction to disappear.

> It is important to observe that the vicious-circle principle is not in itself the solution of vicious-circle paradoxes, but merely the result which a theory must yield if it is to afford a solution of them. It is necessary, that is to say, to construct a theory of expressions containing apparent variables which will yield the vicious-circle principle as an outcome. It is for this reason that we need a reconstruction of logical first principles, and cannot rest satisfied with the mere fact that the paradoxes are due to vicious circles.[29]

Russell scores a serious point in this game against Poincaré. It is Poincaré's doctrine of the universe of discourse that he attacks. There is no single universe of discourse. Even a statement like "for every x, $x = x$" cannot be understood as quantifying over all objects without restriction. The vicious circle principle, as it is proposed by Poincaré, itself violates this principle (the definition of E refers to E). Russell's proposition is a profound reformation of the procedures that we consider legitimate for introducing *new definitions* of objects.

Note the extent of the distance traveled since the *Principles*. The richness of Russell's ontology in 1903 allowed him to advance to a less trivial theory of definitions. The discovery of the contradictions pushed him towards a serious examination of the structure of language. This led in return to the discovery of his theory of descriptions, which would permanently destroy his naiveté. However, along with the discovery his entire ontology would have to be modified.

> The vicious circles arise when a phrase containing such words as *all* or *some* (i.e. containing an apparent variable) appears itself to stand for one of the objects to which the words *all* or *some* are applied. This appearance must, therefore, be deceptive. The difficulty is that there is reason to hold that *all* must be capable of meaning *absolutely all*; thus the phrases in question must not stand for entities at all.[30]

Poincaré would not really respond to this text of Russell's. In 1909 he published his final text on the subject of unpredictability in the journal *Revue de métaphysique et de morale*: "La logique de l'infini". He meticulously reworked his vicious circle theory by applying it to classifications. We cannot classify objects that do not yet exist. However "any definition is a classification".[31] He does not resume the discussion with Russell but instead simply repeats his earlier claims. He puts an end to any future discussion by repeating his profession of faith regarding the role of intuition.

> Mr Russell will doubtless say to me that it is not about psychology, but about logic and epistemology; and I shall be led to answer that there is no logic and epistemology independent from psychology; and this declaration of principles will probably close the discussion because it will bring to light an irreparable difference of view.[32]

2. LEŚNIEWSKI

Leśniewski probably never heard about the arguments between Russell and Poincaré. However, the most interesting logical/philosophical solution to the contradiction appears in his work. In his paper "Leśniewski's Analysis of Russell's Paradox", Boleslaw Sobociński presents the way Leśniewski's systems provide a solution to the contradictions in set theory. Leśniewski's systems were, in fact, introduced to avoid contradictions in the foundation of mathematics in what Leśniewski called a *natural way*. It is difficult to know exactly what it is to be a natural way to avoid contradictions. However, we know what it is, for him, to be a non-natural way to avoid contradiction because we have two examples: Russell's way and Zermelo's way. Here is a quotation about Russell and Zermelo from Sobociński.

> It seems to us that the weakness of these attempts lies in the fact that they are generally limited to modifications of foundations which outlaw the formulation of the known paradoxes within the system. But such a procedure is no protection against unanticipated paradoxes which might appear in the system.[33]

In other words, Russell and Zermelo provided not only *ad hoc* solutions but also solutions directed only against the known contradiction without any idea of what would happen if another kind of contradiction were to appear somewhere else in the foundations of mathematics.

Leśniewski, and following him, Sobociński, rejected set theory, more precisely, rejected set theory as based on the use of "$\in$" in

favour of a new theory, called "Ontology", based on a new connective "ε" which differs from the former in a number of ways.

We will see that the meaning of "ε" is not "user friendly" and that the temptation to go back to naïve set theory in metalinguistic contexts is irrepressible.

> This system, which differs in many ways from contemporary systems, is non contradictory (which is easy to prove), and is an adequate base for the contemporary mathematics. However, it is not very easy to get the system, nor is it easy to penetrate the psychology from which it arose – what precisely were Leśniewski's thoughts about Russell's paradox.[34]

Curiously, there is no suggestion of a definition of "$a \in b$" inside Ontology, a definition that would save the smart naïve properties of "$\in$" but leave the system free of contradiction. In what follows, I will first give a rough sketch of Ontology and describe the essential features of protothetics, which is the logical system in which Ontology is expressed. I will then present Sobociński's solution of the paradox (which is presented as a formalization of Leśniewski's solution).

Next, I will introduce some definitions of simple operators and I will then provide a definition of "$a \in b$". Finally, I will discuss some properties of this "$\in$" which are in fact common properties of the various "$\in$".

2.1. Ontology

Leśniewski's solution to the problem of the paradoxes is not a solution within some version of set theory. Leśniewski provides a system, a calculus of names, which is consistent and in which any attempt to construct statements that violate the vicious circle principle turn out to be false. This calculus of names is what he called "Ontology". As I said above, Ontology is expressed in a very basic logical language, the system of Protothetics. Protothetics is a kind of higher order propositional logic with denumerably many variables of any category and only two primitives: the universal quantifier and identity.

Protothetics is a generalized calculus of propositions containing variables of arbitrary syntactic categories definable starting with the basic category S of sentences.

Definition 1.1

(i) S is a syntactic category

(ii) If $X, X_1, \ldots, X_n$ are syntactic categories, $X/X_1 \ldots X_n$ is a syntactic category.

(iii) Nothing else is a syntactic category.

The wff's of Protothetics are[35]

(i) A variable of type S;

(ii) Identity statements $\ulcorner A \equiv B \urcorner$ where A and B are wff's;

(iii) Generalization: $\llcorner v_1 \ldots v_n \lrcorner \ulcorner A \urcorner$ where A is a wff;

(iv) All the expressions $N(v_1 \ldots v_n)$ where N is introduced by a definition. The general form of these definitions is

$\llcorner v_1 \ldots v_n \lrcorner \ulcorner N(v_1 \ldots v_n) \equiv A(v_1 \ldots v_n) \urcorner$

where N is a new constant and $N(v_1 \ldots v_n)$ is of category S and $A(v_1 \ldots v_n)$ is an already defined wff and $\llcorner v_1 \ldots v_n \lrcorner$ is universal quantification over $v_1 \ldots v_n$.

This particularity of the system keeps it free from contradiction. The rules used to introduce new terms cannot produce contradictions and this is quite independent of the other features of Leśniewski's system.

> The problem of definition in the theory of deduction lies quite outside my system of foundations of mathematics. What interested me in this problem, if I may so express myself, was its own *constructive appeal* – in view of the still rather stepmotherly treatment of it even in the current scientific trend in theory of deduction and theory of theory of deduction.[36]

Leśniewski provided a set of rules for the introduction of new constants that protects any theory using it from any contradiction.

Here are some examples of definitions in Protothetics:

p where p is a propositional variable

$\llcorner p \lrcorner \ulcorner p \urcorner$(which can be used as falsity $\bot$)

$\llcorner p \lrcorner \ulcorner (p \equiv p) \urcorner$ (which can be used as truth T)

$\llcorner p \lrcorner \ulcorner (\neg(p) \equiv (p \equiv \llcorner q \lrcorner \ulcorner q \urcorner)) \urcorner$ (this wff introduces negation)

$\llcorner pq \lrcorner \ulcorner (\wedge pq) \equiv \llcorner f \lrcorner \ulcorner (p \equiv (\llcorner r \lrcorner \ulcorner (p \equiv f(r)) \urcorner \equiv \llcorner r \lrcorner \ulcorner (q \equiv f(r)) \urcorner) \urcorner) \urcorner$ (this wff introduces conjunction).[37]

In the last example, f is of category S/S.

$\llcorner pq \lrcorner \ulcorner ((p \vee q) \equiv \neg(\neg p \wedge q)) \urcorner$

$\llcorner pq \lrcorner \ulcorner ((p \supset q) \equiv (\neg p \vee q)) \urcorner$

There are many equivalent ways to axiomatize Protothetics. Here is a system taken from Słupecki 1950:

A1. $\llcorner pqr \lrcorner \ulcorner \ulcorner \ulcorner p \equiv q \urcorner \equiv \ulcorner \ulcorner r \equiv q \urcorner \equiv \ulcorner p \equiv r \urcorner \urcorner \urcorner \urcorner$

A2. $\llcorner pq \lrcorner \ulcorner \ulcorner \ulcorner p \equiv q \urcorner \equiv \llcorner f \lrcorner \ulcorner f(p) \equiv f(q) \urcorner \urcorner \urcorner$

A3. $\llcorner pq \lrcorner \ulcorner \ulcorner p \equiv q \urcorner \equiv \ulcorner \llcorner f \lrcorner \ulcorner \ulcorner f(p) \equiv f(q) \urcorner \equiv \ulcorner p \equiv q \urcorner \urcorner \urcorner \urcorner$

A4. $\llcorner f \lrcorner \ulcorner (f(\llcorner p \lrcorner \ulcorner p \urcorner) \equiv ((f(\llcorner p \lrcorner \ulcorner p \urcorner) \equiv \llcorner p \lrcorner \ulcorner p \urcorner) \equiv (\llcorner q \lrcorner \ulcorner f(\llcorner p \lrcorner \ulcorner p \urcorner) \equiv f(q) \urcorner))) \urcorner$.

If we add the following 5 rules

R1. Substitution

R2. Detachment

R3. Distribution of quantifiers

R4. Extensionality (of any expression of any category)

R5. Rule of definition (every definition as above is a theorem)

We have a complete system in the following sense: every closed wff of category S is either a theorem or its negation is a theorem. In particular, all the valid inferences in classical logic are valid in protothetics.

It is worth saying a few words about the distinction between Protothetics and the theory of propositional types. Firstly, Protothetics is purely nominalistic: there is no formal semantics in terms of a hierarchy of functions built on truth values. For sure, we have an implicit semantics: the theorems are taken as the true statements, expressions of the type S/S can be seen as denoting one place propositional functions, etc. But these considerations play no role in the theory. Protothetics is a syntactic theory and thus nominalistic. This brings me to a second remark. A system of Protothetics is never completely developed. One can always introduce a new constant and thus new theorems. Each system is complete but this completeness is relative to the constant functors already introduced, so each system is a work in progress. This is the major difference from propositional type theory where the hierarchy of propositional functions is given once and for all.

Ontology is an extension of Protothetics obtained by adding a second basic category, the category N of names.

Definition 1.2

(i) S and N are syntactic categories;

(ii) If $X, X_1, \ldots, X_n$ are syntactic categories, $X/X_1 \ldots X_n$ is a syntactic category.

(iii) Nothing else is a syntactic category.

Ontology contains a new constant "ε" of category S/NN. $A\varepsilon B$ should be read A is a part of B. "ε" should not be confused with the "$\in$" of set theory. For example, "ε" is transitive whereas "$\in$" is not. Equality between expressions of category N is introduced in the following way:

$$\llcorner abc \lrcorner \ulcorner (a = b) \equiv \ulcorner (c\varepsilon a) \equiv (c\varepsilon b)\urcorner\urcorner$$

A system for Ontology is obtained by adding

A5. $\llcorner ab \lrcorner \ulcorner (a\varepsilon b) \equiv ((\neg(\llcorner c \lrcorner \neg(c\varepsilon a)) \wedge \llcorner dc \lrcorner (((d\varepsilon a) \wedge (c\varepsilon a)) \supset (d\varepsilon c)) \wedge \llcorner d \lrcorner ((d\varepsilon a) \supset (d\varepsilon b)))\urcorner$

We can prove that the truth conditions of $a\varepsilon b$ are:

> $a\varepsilon b$ is true iff a is a name of an individual (a denotes one and only one thing) and a is among the b.

So a name can denote an individual as well as a multiplicity. If B denotes a multiplicity x *and* y *and*... *and* z we will say that b is $x, y, \ldots, z$. This is a metanotation: $x\varepsilon b, y\varepsilon b$, etc. have some meaning but not $x\varepsilon x, y, \ldots, z$. Examples:

> "Socrates is a man" is true.
>
> "Socrates is Socrates" is true.
>
> "Animals are animals" is false (literally, but "All animals are animals" is true). In fact, if A stands for the animals, $A\varepsilon A$ is false but $\llcorner x \lrcorner \ulcorner (x\varepsilon A \supset x\varepsilon A)\urcorner$ is true.
>
> "Hamlet is the hero of a tragedy by Shakespeare" is false (because Hamlet does not exist).

A5 states that $a\varepsilon b$ is the conjunction of three properties:

(1) $\llcorner \exists y \lrcorner (y\varepsilon a)$

(2) $\llcorner yz \lrcorner ((y\varepsilon x \wedge z\varepsilon x) \supset y\varepsilon z)$

(3) $\llcorner y \lrcorner (y\varepsilon x \supset y\varepsilon b)$

From (2) and the definition of identity $\llcorner abc \lrcorner \ulcorner (a = b) \equiv \ulcorner (c\varepsilon a) \equiv (c\varepsilon b)\urcorner\urcorner$, we easily show that $a\varepsilon b$ means that (i) something is (an) a, (ii) any two things which are a are identical and (iii) every thing which is an a is also a b.[38]

2.2. The contradiction in Leśniewski's Ontology

In order to show how Russell's paradox cannot arise in Ontology, Sobociński introduced the distinction between a class as one (collective class) and a class as many (distributive class). Let a be an object. $Kl(a)$ is an object called the *collective class* of a. Conversely, if B is a collective class, then $el(B)$ is an object called the *distributive class* of B. The two notions are related by the following equivalence:

$$\llcorner ab \lrcorner (b\varepsilon el(a) \equiv \llcorner \exists c \lrcorner (a\varepsilon Kl(c) \wedge b\varepsilon c))$$

Let us illustrate it by the following example. Let S stand for Socrates and H stands for humanity. Both are names of individuals. Let us instantiate the above equivalence by letting a be H and b be S. We have

$$(S\varepsilon el(H) \equiv \llcorner \exists c \lrcorner (H\varepsilon Kl(c) \wedge S\varepsilon c))$$

There is certainly such a c, the multiplicity of men. In other words, Socrates is one of the elements of the collective class of men iff there is a distributive class c such that humanity is the collective class of c and Socrates is one of the c.

A limiting case is when the distributive class is an individual. In that case the collective class and the distributive class collapse into the same object, the individual. For example, let us consider Socrates:

$$(S\varepsilon el(S) \equiv \llcorner\exists c\lrcorner(S\varepsilon Kl(c) \wedge S\varepsilon S))$$

taking S as c this is equivalent to

$$(S\varepsilon S \equiv (S\varepsilon S \wedge S\varepsilon S)).$$

The following properties of el and Kl can be established.[39]

K1 $\llcorner a\lrcorner(\llcorner\exists b\lrcorner(b\varepsilon a) \equiv \llcorner\exists c\lrcorner(c\varepsilon Kl(a)))$

There are some a iff there is a collective class of the a's.

K2 $\llcorner abc\lrcorner((a\varepsilon Kl(c) \wedge b\varepsilon Kl(c)) \supset a = b)$

If two objects are the collective class of the c, they are the same object.

K3 $\llcorner a\lrcorner(a\varepsilon a \supset a = Kl(a))$

Any individual is the collective class of this individual.

K4 $\llcorner ab\lrcorner((a\varepsilon Kl(b) \wedge b\varepsilon b) \supset a = b)$

If an individual a is the collective class of the b's and b is an individual, then a is b.

K5 $\llcorner ab\lrcorner(a\varepsilon Kl(b) \equiv (a = Kl(b)))$

If a is one of the objects "collective class" of the b's then a is the collective class of the b's.

K6 $\llcorner ab\lrcorner(a\varepsilon Kl(b) \equiv (a\varepsilon Kl(Kl(b))))$

K7 $\llcorner abc\lrcorner((a\varepsilon Kl(b) \wedge a\varepsilon Kl(c)) \supset (Kl(b) = Kl(c)))$

K8 $\llcorner ab\lrcorner\ulcorner(a\varepsilon el(b) \equiv (\llcorner\exists c\lrcorner(b\varepsilon Kl(c) \wedge a\varepsilon c))\urcorner$

a is one of the objects of the distributive class $el(b)$ iff b is the collective class of some c and a is a c.

K9 $\llcorner ab\lrcorner\ulcorner a\varepsilon Kl(el(Kl(b))) \equiv a\varepsilon Kl(b)\urcorner$

K10 $\llcorner a\lrcorner(a\varepsilon a \supset a\varepsilon el(a))$

If a is an individual, a is one of the distributive class of the a's or each individual is an element of itself.

K11 $\llcorner abc\lrcorner((a\varepsilon el(b) \wedge b\varepsilon el(c)) \supset a\varepsilon el(c))$

K12 $\llcorner ab\lrcorner(a\varepsilon el(b) \supset b\varepsilon b))$

K13 $\llcorner ab\lrcorner((a\varepsilon el(b) \wedge b\varepsilon el(a)) \supset a = b)$

K14 $\llcorner ab\lrcorner\ulcorner a\varepsilon Kl(b) \equiv (a\varepsilon a \wedge \llcorner\exists c\lrcorner c\varepsilon b \wedge (\llcorner c\lrcorner c\varepsilon b \supset c\varepsilon el(a))) \wedge$
$(\llcorner c\lrcorner(c\varepsilon el(a) \supset \llcorner\exists de\lrcorner(d\varepsilon b \wedge e\varepsilon el(d) \wedge e\varepsilon el(c)))\urcorner$

Sobociński showed that the two axioms used by Russell[40] and others

A1 $\llcorner a\lrcorner\llcorner\exists b\lrcorner(b\varepsilon Kl(a))$

A2 $\llcorner abcd\lrcorner((a\varepsilon Kl(c) \wedge a\varepsilon Kl(d) \wedge b\varepsilon d) \supset b\varepsilon c)$ [41]

are false (their negations can be proved) and there is no more contradiction...

> The distinctions made by Leśniewski, enable him to realize that the illusory intuitiveness of A2 [...] is the consequence of the confusion caused by the use of the same noun for the two different concepts [distributive class and collective class].[42]

The question now is: is it possible to define '$\in$', i.e. to define a relation that possesses most of the properties of the naïve $\in$ but free of contradiction?

2.3. Set theory

We will need some definitions.
Definition of the empty name Λ:

D1 $\llcorner x\lrcorner\ulcorner x\varepsilon\Lambda \equiv (x\varepsilon x \wedge \neg(x\varepsilon x))\urcorner$ [43]

A consequence of this definition is that $x\varepsilon\Lambda$ is always false of x. Λ does not denote, it fails to denote.
Definition (there is at least one x)

D2 $\llcorner x \lrcorner \ulcorner Ex(x) \equiv [\exists y](y\varepsilon x)\urcorner$

Definition (there is at most one x)

D3 $\llcorner x \lrcorner \ulcorner Mo(x) \equiv \llcorner yz \lrcorner ((y\varepsilon x \wedge z\varepsilon x) \supset y\varepsilon z)\urcorner$

Definition (ontological inclusion)[44]

D4 $\llcorner xy \lrcorner \ulcorner x \ \subseteq^* y \equiv \llcorner z \lrcorner (z\varepsilon x \supset z\varepsilon y)\urcorner)$

Here is a sentence equivalent to the axiom of Ontology.

T1 $\llcorner xy \lrcorner \ulcorner x\varepsilon y \equiv (E(x) \wedge Mo(x) \wedge x \ \subseteq^* y)\urcorner$

Some corollaries

C1 $\llcorner x \lrcorner (\neg(x\varepsilon\Lambda))$ and $\llcorner x \lrcorner (\neg(\Lambda\varepsilon x))$

T2 Transitivity of ε

$\llcorner xyz \lrcorner ((x\varepsilon y \wedge y\varepsilon z) \supset x\varepsilon z)$

T3 Reflexivity of $\subseteq^*$

$\llcorner xy \lrcorner (x\varepsilon y \supset x\varepsilon y)$ and thus $\llcorner y \lrcorner (y \ \subseteq^* y)$

Some properties of "ε" are properties of "$\in$" but some are not. So the question is: can we define some "$\in$" in Ontology?[45] Beforehand we have to decide the status of russellian sets in Ontology. Are they individuals, i.e., can a sentence like $x\varepsilon y$ be possibly true, or are they multiplicities? Our naïve conception of sets says that they are both: sets have members and can be members, and from some point of view, this is the origin of the contradiction. However, we know that Ontology is consistent and that following the rules for the definitions cannot generate a contradiction. The following definition seems to be adequate.

2.4. Inductive definition of Russellian Classes (RC)

Let us call a name *primitive* if it belongs to pure Ontology.
We define the hierarchy of Russellian Simple Sets (RSS).

(i) For name a, the function $\{\}$ introduce a new name noted $\{a\}$ called singleton a. If a is primitive (a is not already an image of $\{\}$, $\{a\}$ is a RSS *of type* 1. $\{a\}$ is such that $\{a\}\varepsilon\{a\}$. $\{a\}$ is new in the sense that $\{a\} \neq b$ for every primitive b. Because, $\{a\}\varepsilon\{a\}$, $\{a\}$ is an individual.
If a is a russellian set of type n, then $\{a\}$ is a RSS of type $n+1$.
It is clear that no term of type n is identical to a term of type $n+1$.

(ii) Let $a_0, \ldots, a_i, \ldots$ be a sequence of RSS such that $a_i = \{y_i\}$ for each i.
$\bigcup_i a_i = \{y_0, \ldots, y_j, \ldots\}$ is a RC of the same type as the greatest type of the a_i's. If there is no greatest type in the sequence, $\bigcup_i a_i$ is not defined.
Remark $\bigcup_i a_i$ is of the form $\{z\}$ were $y_i\varepsilon z$ for all and only the y_j of the sequence $y_0, \ldots, y_j, \ldots$.

(iii) Let $x_0, \ldots, x_i, \ldots$ be a sequence of RC such that $x_i = \{z_i\}$ for each i.
$\bigcup_i x_i = \{z_0, \ldots, z_j, \ldots\}$ is a RC of the same type as the greatest type of the x_i's. If there is no greatest type in the sequence, $\bigcup_i x_i$ is not defined.
The last restriction is necessary to block the definition of a teratological object like $\{\{\{\ldots\{\{\{a\}\}\}\ldots\}\}\}$ which has no type. In general, a sequence of unbound sets *does* not give a set by applying (ii) or (iii).

Example:

Suppose we start with a, b and c.

Using (i) we can define $\{a\}, \{b\}, \{c\}, \{\{a\}\}, \{\{b\}\}, \{\{c\}\}$, *etc.*

Using (ii), we can, for example, define $\{d\}$ where d is the multiplicity $a, b, \{c\}$. If a, b, c are of type 0, $\{d\}$ is of type 2.

Using (iii), we can, for example, define $\{e\}$ where e is the multiplicity $a, b, \{\{a\}, \{b, c\}\}, \{\{\{a\}\}\}$. $\{e\}$ is of type 4.

However, the sequence $\{a\}, \{\{a\}\}, \{\{\{a\}\}\}, \ldots$ of sets does not give a set by applying (iii).

We see that we can define individuals that look like classical sets. But do they have properties of classical sets?

Definition of "∈"

$$\llcorner xy \lrcorner \ulcorner x \in \{y\} \equiv (x \varepsilon y) \urcorner$$

Definition of ∅

$$\varnothing = \{\Lambda\}$$

Proposition

$$\llcorner x \lrcorner (\neg(x \in \varnothing))$$

Proposition

$$\llcorner x \lrcorner (\neg(x \in x))$$

One of the necessary conditions for $x \in y$ to be true is that the type of y is strictly greater than the type of x, so $x \in x$ can never be true.

Extensionality can be easily proved. Here is a sketch:

> $\ulcorner z \in x \equiv z \in y \urcorner$ is equivalent to $\ulcorner z \in \{a\} \equiv z \in \{b\} \urcorner$ which is equivalent to $\ulcorner z \varepsilon a \equiv z \varepsilon b \urcorner$ which is equivalent to $a = b$.

Foundation is also easy to prove. Here is a sketch:

> Suppose that x is not empty. We have to prove that $\exists y (y \in x \wedge \neg \exists z (z \in y \wedge z \in x))$

As x is not empty, we take as y one that has the lowest type. If there is a z such that $z \in y$, z has a lower type than y which contradicts that $z \in x$ because y is of the lowest type in x.

Union axiom and power set axiom are also sound definitions of new sets when applied to existent sets.

3. TENTATIVE CONCLUSION

Russell's contradiction can be naturally eliminated in a theory like Ontology, which does not use the notion of class. Curiously, such a

theory is possible only because we have a clear notion of what is a sound definition, the notion which was at the heart of the Poincaré-Russell quarrel. Moreover, it seems possible to define a rather naïve set theory which is free of contradiction (as any other constructions in Ontology) by definition in Ontology. This suggests that the very problem about the contradiction was really a problem about how to understand and restrict definitions as Poincaré thought.

However, a lot of questions remain unanswered about reintroducing set theory in Ontology. One of them is the following: is it possible to define a formal semantics that is in harmony with the spirit of Ontology? And which are the axioms for this set theory? This will be the object of another paper.

Notes

[1] *The Principles*, p. 3.

[2] « Les Mathématiques et la logique » p.316.

[3] *The Principles*, p. 43. Note that the *Principles* and Russell's other texts from this period contain within them a double ambiguity or systematic confusion, which sometimes makes them difficult to interpret. Russell uses the semantic tool of expressions in order to designate what expressions refer to. The use of the expression "term" is a particularly flagrant case and the passage cited above is particularly troublesome. The terms are entities that are used to compose propositions, which are themselves sometimes statements and sometimes that to which the statements refer.

[4] Russell had probably already started his lectures on Meinong from which he would publish a number of important papers between 1899 and 1907. Three of these papers appeared in *Essays in Analysis*. See also Douglas Lackey's introduction to the volume "Russell's Critique of Meinong".

[5] Printed in the *Essays*

[6] *The Principles*, p. X.

[7] *The Principles*, p. 451.

[8] *The Principles*, p. 113 .Russell deals with this subject in the first chapter of the second part of the *Principles* entitled *Numbers*. Russell makes no reference to the contradiction that brings him to his principle of unrestricted extensionality, which was, curiously, the subject of the preceding chapter.

[9] *The Principles*, p. 113.

[10] *The Principles*, p. 114.

[11] *The Principles*, p. 115.

[12] *The Principles*, p. 115.

[13] *The Principles*, p. 497.

[14] This has nothing to do with the fact that Russell admits in the introduction to the second edition that his theory of descriptions is one of two discoveries that led him to reject the Platonism of the *Prinicples*. The second is the rejection of

classes! See *The Principles*, p. X.
[15] *The Principles*, p. 63.
[16] Review of *Science and Hypothesis*, p. 412
[17] *La science et l'hypothèse* p. 31.
[18] *La science et l'hypothèse* p. 39.
[19] *La science et l'hypothèse* p. 41.
[20] *La science et l'hypothèse* p. 41.
[21] "On Some Difficulties in the Theory of Transfinite Numbers and Order Types", *Proceedings of the London Mathematical Society*, Series 2, **4**, 1906, pp. 29-53. Reprint in *Essays in Analysis*, pp. 135-164. The references refer to this latest edition
[22] *Essays in Analysis*, p. 142.
[23] See*Essays in Analysis*, p. 147.
[24] *Essays in Analysis*, p. 164.
[25] *Essays in Analysis*, p. 155.
[26] « Les mathématiques... » p. 309.
[27] « Les paradoxes... », p. 627. The original manuscript is reprint in the *Essays...* pp. 190-214 with its original title: "On 'Insolubilia' and their Solution by Symbolic Logic". Quotation from p. 190.
[28] « Les paradoxes... », p. 633, *Essays...* p.197.
[29] « Les paradoxes... », p. 640-641, *Essays...* p.197.
[30] « Les paradoxes... », p. 648-649, *Essays...* p.197. The French version contains a translation error when compared to the manuscript published in the *Essays*. The French term "phrase", which appears twice, is a bad translation for the English term "phrase".
[31] « La logique de l'infini », p.402.
[32] « La logique de l'infini », p.414.
[33] "Leśniewski's Analysis of Russell's Paradox", *Leśniewski's Systems: Ontology and Mereology*, Jan T. J. Srzednicki *et als* (eds), Martinus Nijhoff Publishers, The Hague, 1984 (1949-1950).
[34] « Leśniewski's analysis », p. 11.
[35] For the sake of simplicity, I won't use Polish notation.
[36] Leśniewski S., "On Definitions in the So-called Theory of Deduction", *Stanislaw Leśniewski Collected Works*, Vol II, Surma S. J. *et als* (eds.).
[37] Most of these definition were introduced by Tarski who was the only doctoral student of Leśniewski. See "On the primitive term of logistic", in *Logic, Semantics, and Metamathematics*, J. Corcoran (ed.), Hackett Publishing, 1983, pp. 1-23.
[38] See J. Słupecki, « S. Leśniewski Calculus of Names", in *Studia Logica*, III, 1955, reprint in *Leśniewski's Systems: Ontology and Mereology* pp. 59-122
[39] See Sobociński, pp.33-34.
[40] Here, we should understand that ε is interpreted as $\in$.
[41] A1 says that for any multiplicity, there is a collective class of this multiplicity (comprehension). A2 says that two different multiplicities give raise to two different collective classes (extensionality).
[42] Sobociński, pp.30.
[43] There is no fundamental difference berween definition and thesis for Leśniewski.

This formula is a definition because the left member contains a new term.
[44] We need a new symbol to escape confusion with set theoretic inclusion.
[45] Our proposition is similar, in some aspects, to Peters Simons', "On Understanding Leśniewski", *History and Philosophy of Logic*, 3, 1982, pp. 165-191. (see section 9 starting on p. 188). The main differences are firstly that, in Simons approach, classes belong to a new category, "class name". If x is a name and y is a name of a class, $x = y$ is neither true or false because it is not a wff. Secondly, "$\in$" is a primitive and finally, his approach seems to make no natural room for mixed classes, i.e., classes having members of different types like, for example, $\{a, \{a, b\}\}$.

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