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GAMES AND LOGIC

ABSTRACT: The idea behind these games is to obtain an alternative characterization of logical notions cherished by logicians such as truth in a model, or provability (in a formal system). We offer a quick survey of Hintikka's evaluation games, which offer an alternative notion of truth in a model for first-order langauges. These are win-lose, extensive games of perfect information. We then consider a variation of these games, IF games, which are win-lose extensive games of imperfect information. Both games presuppose that the meaning of the basic vocabulary of the language is given. To give an account of the linguistic conventions which settle the meaning of the basic vocabulary, we consider signaling games, inspired by Lewis' work. We close with IF probabilistic games, a strategic variant of IF games which combines semantical games with von Neumann's minimax theorem.[1]

1. EVALUATION GAMES: GAME-THEORETICAL SEMANTICS

Evaluation games for first-order logic have arisen from the work of Henkin in the 50's and the work of Hintikka in the 70'. For Hintikka the purpose is to give a game-theoretical characterization of truth and falsity for first-order sentences. The driving force here are the standard quantifiers, the universal "for every x", and the existential "for some

x". Their scopal dependencies and independencies in a first-order sentence are recast in terms of the strategic interaction of two players in a game of perfect information. For instance, the first-order sentence $\forall x \exists y B(x,y)$ is analyzed in terms of a game between the universal player (Abelard) and the existential player (Eloise). They choose individuals from the underlying universe of discourse to be the values of the variables x and y, respectively. The order of the choices and the information sets of the players are indicated by the syntax of the sentence. Relative to the universe, the sentence is true if and only if there is a method (a strategy) for Eloise that produces a choice b for every choice a by Abelard so that $B(a,b)$ holds. And the sentence is false if there is a way for Abelard to choose an individual a so that for every possible choice b by Eloise, $B(a,b)$ does not hold.

When the dependencies and independencies between quantifiers become more complex, the interaction between the two players becomes less trivial. For instance, consider the game associated with the first-order sentence

$$\forall x \exists y \forall x' \exists y' Q(x,x',y,y').$$

A strategy for Eloise will have to give her, first, a choice b for every individual a chosen by her opponent, and, second, another choice b' for each individual a and a', chosen by Abelard. But for the second choice, the strategy may use information encoded in her first choice for b. The game-theoretical view connects in a nice way the dependencies and independencies of quantifiers with the choice functions which codify the strategies of the two players in semantical games. This view of quantification is at "the heart of how classical logicians in the twenties viewed the nature of quantification." (Goldfarb 1979).

1.1. Semantical games: extensive games of perfect information

Let us make the above informal notions more precise. A semantical game is played by two players, $\exists$ (Eloise) and $\forall$ (Abelard). They consider a first-order formula φ in negation normal form, model $\mathbb{M}$, which interprets the vocabulary of φ, and an assignment s in $\mathbb{M}$ which includes the free variables of φ. $\exists$ tries to show that φ is true in $\mathbb{M}$ (relative to the assignment s), and $\forall$ tries to show that φ is false in $\mathbb{M}$. The game starts from the initial position (φ, s), and after each move the players reach a position (ψ, r), where ψ is a subformula of φ and r is an assignment which eventually extends s. Here are the rules of the game:

(1) (ψ, r), where ψ is a literal. The game stops. If $\mathbb{M}, r \models \psi$ then $\exists$ wins. Otherwise $\forall$ wins. Notice that the notion $\mathbb{M}, r \models \psi$ has been defined.

(2) $(\psi \vee \psi', r)$: $\exists$ chooses $\theta \in \{\psi, \psi'\}$ and players move to (θ, r).

(3) $(\psi \wedge \psi', r)$: $\forall$ chooses $\theta \in \{\psi, \psi'\}$ and players move to (θ, r).

(4) $(\exists x \psi, r)$: $\exists$ chooses $a \in M$ and players move to $(\psi, r(x/a))$.

(5) $(\forall x \psi, r)$: $\forall$ chooses $a \in M$ and players move to $(\psi, r(x/a))$.

This game, which we denote by $G(\mathbb{M}, s, \varphi)$, may be easily reformulated as a finite two-player, win-lose extensive game with perfect information, $G(\mathbb{M}, s, \varphi) = (N, H, Z, P, (u_p)_{p \in N})$, where

- N is the set of players, $N = \{\exists, \forall\}$,
- H is the set of histories of the game, with one initial root, (s, φ)
- Z is the set of maximal histories (plays),
- $P : H - Z \rightarrow N$ is the player function which tells whose player's turn is to move, and finally
- u_p: $Z \rightarrow \{0, 1\}$ is the payoff function for player p such that for each $h \in Z$: $u_p(h) = 0$ or $u_p(h) = 1$ (but not both).

The last condition explicitates the win-lose property of the game. When $u_p(h) = 1$ we say that p wins the play h; and when $u_p(h) = 0$, we say that p looses h.

When making choices as prescribed by the rules of the game, each player follows a (deterministic) strategy which gives him or her the next move to make. A strategy for player p in the game $G(\mathbb{M}, s, \varphi)$ is a function σ_p defined on all (non-maximal) histories h where p is to move; $\sigma_p(h)$ is the next position to be reached in the game. The strategy σ_p is winning if p wins every maximal history (play) where he or she follows σ_p.

Game-theoretical truth, $\mathbb{M}, s \models_{GTS}{}^{+} \varphi$, is defined as the existence of a winning strategy for Eloise; and game-theoretical falsity, $\mathbb{M}, s \models_{GTS}{}^{-} \varphi$ is defined as the existence of a winning strategy for Abelard.

The principle of determinacy, known as Zermelo Theorem (1913), ensures us that the principle of bivalence holds: Every win-lose game with finite horizon and one initial root is determinate.

1.2. Game-theoretical negation

We relax the assumption that negation occurs only in front of atomic formulas. We thus need a game-rule for negation. It is given in terms of the players "switching roles". We reformulate the rules of the game to make place for roles, Verifier and Falsifier:

- At the beginning of the game $\exists$ is the Verifier (V) and $\forall$ is the Falsifier (F)

The rules of the game $G(\mathbb{M}, s, \varphi)$ are now restated as:

- $(P(t_1, \ldots, t_n), r)$: The game stops. If r satisfies $P(t_1, \ldots, t_n)$, then V wins. Otherwise F wins.
- $(\neg\psi, r)$: Players move to (ψ, r) with roles inversed.
- $(\psi \vee \psi', r)$: V chooses $\theta \in \{\psi, \psi'\}$ and players move to (θ, r).
- $(\psi \wedge \psi', r)$: F chooses $\theta \in \{\psi, \psi'\}$ and players move to (θ, r).
- $(\exists x \psi, r)$: V chooses $a \in M$ and players move to $(\psi, r(x/a))$.
- $(\forall x \psi, r)$: F chooses $a \in M$ and players move to $(\psi, r(x/a))$.

Obviously we are still inside the class of finite, 2-player, win-lose extensive games of perfect information. The notion of strategy for a player p is defined exactly as before, as are game-theoretical truth and falsity. Zermelo's theorem still applies. The switching role interpretation of negation makes possible the following fact for every first-order formula φ, model $\mathbb{M}$ and assignment s in $\mathbb{M}$ whose domain contains $Free(\varphi)$:

$$
\begin{array}{ccc}
\mathbb{M}, s \models_{GTS}{}^{+} \neg\varphi & \text{iff} & \mathbb{M}, s \models_{GTS}{}^{-} \varphi \\
\mathbb{M}, s \models_{GTS}{}^{-} \neg\varphi & \text{iff} & \mathbb{M}, s \models_{GTS}{}^{+} \varphi.
\end{array}
$$

Using Zermelo's theorem we can show that our game game-theoretical negation is contradictory negation: $\mathbb{M}, s \vDash_{GTS}{}^{+} \neg\varphi$ iff $\mathbb{M}, s \nvDash^{+} \varphi$.

It can also be shown that we recover Tarski type satisfaction relation:

Theorem. (Hodges 1997; Mann et al. 2011) Let φ be a first-order formula, $\mathbb{M}$ a suitable model for the language of φ and s an assignment in $\mathbb{M}$ whose domain includes the free variables of φ. Then the following holds:

1. $M, s \vDash_{GTS}{}^{+} \neg\varphi$ iff $M, s \nvDash_{GTS}{}^{+} \varphi$

2. $M, s \vDash_{GTS}{}^{+} (\varphi \wedge \psi)$ iff $M, s \vDash_{GTS}{}^{+} \varphi$ and $M, s \vDash_{GTS}{}^{+} \psi$

3. $M, s \vDash_{GTS}{}^{+} (\varphi \vee \psi)$ iff $M, s \vDash_{GTS}{}^{+} \varphi$ or $M, s \vDash_{GTS}{}^{+} \psi$

4. $M, s \vDash_{GTS}{}^{+} \exists x \varphi$ iff $M, s(x/a) \vDash_{GTS}{}^{+} \varphi$ for some $a \in M$

5. $M, s \vDash_{GTS}{}^{+} \forall x \varphi$ iff $M, s(x/a) \vDash_{GTS}{}^{+} \varphi$ for every $a \in M$.

Although the game-theoretical interpretation of first-order logic captures the Tarskian interpretation, we prefer it to the latter for it shows how many equivalences of first-order logic may be interpreted as recipees for converting one winning strategy in one game into a winning strategy in another game in the underlying model. (For further discussions on this point we refer the reader to Mann et al. (2011) and van Benthem (forth.)). Let us illustrate this for the principle of distributivity.

Example Recall the principle of distributivity for FOL:

$$\varphi \vee (\psi \wedge \theta) \equiv (\varphi \vee \psi) \wedge (\varphi \vee \theta)$$

Suppose Eloise has a winning strategy σ in the game $G(\mathbb{M}, s, \varphi \vee (\psi \wedge \theta))$. Suppose the first choice of Eloise according to σ is φ. Define a winning strategy τ for Eloise in $G(\mathbb{M}, s, (\varphi \vee \psi) \wedge (\varphi \vee \theta))$ as follows. If Abelard chooses $\varphi \vee \psi$, then let τ pick up φ and then mimic σ for the rest of the game in φ. If Abelard chooses $\varphi \vee \theta$, then let τ pick up φ and mimic σ for the rest of the game in φ.

Suppose now that the first choice of Eloise according to σ is $(\psi \wedge \theta)$. Then σ must be a winning strategy in both ψ and θ. Define a winning strategy τ for Eloise in $G(\mathbb{M}, s, (\varphi \vee \psi) \wedge (\varphi \vee \theta))$ as follows. If Abelard chooses $\varphi \vee \psi$, then let τ pick up ψ and mimic σ for the rest of the game in ψ. If Abelard chooses $\varphi \vee \theta$, then let τ pick up θ and mimic σ for the rest of the game in θ. The converse is shown in a similar way.

2. EVALUATION GAMES: INDEPENDENCE-FRIENDLY LOGIC

Let us return to our earlier game associated with the first-order sentence

$$\forall x \exists y \forall x' \exists y' Q(x, x', y, y').$$

Suppose that instead of having y' depending on both x and x', we want y' to depend only on x' (and keep the other dependencies intact). Game-theoretically this will correspond to the idea that in her strategies, Eloise will not be allowed to use information about the choice for x but only information about the choice for x' and information about her own earlier choice for y. To re-establish the correspondence between syntax and the information sets of the players in the underlying game, Hintikka & Sandu (1989) introduced a new notation

$$\forall x \exists y \forall x' (\exists y' / \{x, y\}) Q(x, x', y, y')$$

Intuitively: when choosing a value for y' Eloise does not know the choice of her opponent for x nor her own earlier choice for y. The phenomena of the players ignoring some of their own earlier choices is known in the literature as imperfect recall. It makes the playability of the games rather hard. An alternative interpretation is given by Barwise (1979) (in the context of branching quantifiers). We think of the players as forming teams, each occurrence of the universal quantifier or conjunction being one player in Abelard's team, and each occurrence of the existential quantifier or disjunction being a player in Eloise's team. The interaction of the players in the same team gives rise to the phenomenon of *signaling*. Here is one example (Janssen & Dechesne 2006). We show that the IF sentence

$$\forall x_0 \forall x_2 (x_0 \neq x_2 \vee (\exists x_1 / \{x_0\}) x_0 = x_1)$$

is a logical truth by showing that the team of Eloise, consisting of two players, has a winning strategy on every model (set) $\mathbb{M}$ which has at least two individuals:

- Let Abelard choose a_0 and a_2 from $\mathbb{M}$.
- If $a_0 \neq a_2$, then the first player in Eloise team chooses Left. Eloise's team wins.
- If $a_0 = a_2$, then the same player chooses Right and the second player chooses a_2. Eloise's team wins again.

Notice that despite the second player in the team of Eloise not knowing the choice of the first player in Abelard's team, the first player knows it, and signals it to her team mate: choosing Right is a way for her to say "choose the same as the second player in Abelard's team" (the second player in Eloise's team knows that choice).

2.1. *Semantical games: extensive games of imperfect information*

The syntax of IF logic is defined as in ordinary first-order logic, except for the clauses for the quantifiers which have now the form $(\exists x/W)\varphi$ and $(\forall x/W)\varphi$ where W is a finite set of variables. When W is the empty set, we recover the standard quantifiers.

The notions of models, assignments and the satisfaction clauses for atomic formulas are the same as for ordinary first-order languages. The game-theoretical interpretation extends to the present case: the assumption of the players' information in finite 2-player, win-lose extensive games is now relaxed to imperfect information. When φ is an IF formula, $\mathbb{M}$ a model and s an assignment whose domain includes the free variables of φ, the rules of the semantical game $G(\mathbb{M}, s, \varphi)$ are identical with the six rules given earlier: in the last two clauses $(\exists x\psi, r)$ and $(\forall x\psi, r)$ are replaced with $((\exists x/W)\psi, r)$ and $((\forall x/W)\psi, r)$, respectively. The changes will affect only the information sets and thereby the strategies of the players. The imperfect information comes in the form of the players' restricted access to the current assignment in the game. The slash "/" introduces an equivalence relation on the set of histories in the extensive game which are decision points for one and the same player. This equivalence relation is determined in the following way.

Let W be a set of finite variables, s and s' be two assignments in a model $\mathbb{M}$ with the same domain which includes W. We say that that s

and s' are W-equivalent, $s \approx_W s'$, if for every variable $x \in dom(s) - W$ we have $s(x) = s'(x)$.

Every history h in $G(\mathbb{M}, s, \varphi)$ induces an assignment s_h in the model $\mathbb{M}$, which extends or modifies the initial assignment s. One can then define two indistinguishable relations $\sim_\exists$ and $\sim_\forall$ on the set of non-maximal histories of the game. For $\sim_\exists$ there are two cases:

- It is Eloise's turn to move corresponding to a disjunction $\psi \vee \psi'$. In the context of the extensive game, let h and h' be two histories in the game where such a move is supposed to happen. Then we stipulate that
$$h \sim_\exists h' \quad \Leftrightarrow \quad s_h = s_{h'}.$$
- It is Eloise's turn to move corresponding to $(\exists x/W)\psi$. Then for any two histories h, h' where such a move is about to happen, we stipulate:
$$h \sim_\exists h' \quad \Leftrightarrow \quad s_h \approx_W s_{h'}.$$

The relation $\sim_\forall$ is specified completely analogously.

The relations $\sim_\exists$ and $\sim_\forall$ specify exactly how much information the players have at their disposal at a given decision point.

A strategy σ_p for player p in the semantical game $G(\mathbb{M}, s, \varphi)$ is defined exactly as in the games of perfect information, except for the requirement of uniformity:

- for every h, h'
$$h \sim_p h' \quad \Rightarrow \quad \sigma_p(h) = \sigma_p(h').$$

Truth and falsity of an IF formula in a model are defined exactly as before, keeping in mind, of course, that strategies are now uniform.

2.2. *Indeterminacy*

Imperfect information introduces indeterminacy into the logic, as the next example shows.

Example (MatchingPennies) This is a well known game played by two players, who turn secretely a coin to Heads or Tails. The coins are revealed simultaneously. The first player wins if the outcomes match; the second player wins if they differ. We can model this game in IF logic by the sentence $\varphi_{MP} = \forall x(\exists y/\{x\})\, x = y$ and a two element model $\mathbb{M} = \{a, b\}$. In the extensive game the histories

$$\begin{aligned} h_a &= ((\varphi_{MP}, \varnothing), (x, a)) \\ h_b &= ((\varphi_{MP}, \varnothing), (x, b)) \end{aligned}$$

which represent the two possible choices for Abelard, are equivalent for Eloise, i.e., $h_a \sim_{\exists} h_b$. (Note that the corresponding assignments $(x, a)\}$and (x, b) are trivially $\{x\}$ −equivalent). Let σ be a strategy for Eloise. Given the uniformity requirement, Eloise must choose the same value $c \in \{a, b\}$ for y in both cases:

$$\sigma(h_a) = \sigma(h_b) = (y, c)$$

For this strategy to be winning, we must have $a = b = c$ which is impossible. Let τ be a strategy for Abelard such that $\tau(\varphi_{MP}, \varnothing) = (x, c)$. Then τ is a winning strategy iff Abelard wins against any possible move by Eloise, that is, Abelard must win both plays

$$((\varphi_{MP}, \varnothing), (x, c), (y, a))$$
$$((\varphi_{MP}, \varnothing), (x, c), (y, b))$$

For this to happen we must have: $a \neq c$ and $b \neq c$ which is impossible. Thus neither Eloise nor Abelard has a winning strategy in the game.

- One can show in the same way that the Inverted Matching Pennies sentence

$$\varphi_{IMP} = \forall x(\exists y/\{x\})x \neq y$$

is also indeterminate (neither true nor false on all models with at least two elements).

2.3. *Expressive power and playability of games*

The fact that the patterns of dependencies and independencies between quantifiers in IF logic is greater than in ordinary first-order logic allows us to define in the former concepts which are not definable in the latter. Here are two examples.

A function f is an *involution* if $f(f(x)) = x$ for all x in its domain. We say that x is a *fixed point* for the function f if $f(x) = x$. A finite structure has an *even* number of elements if and only if there is a way of pairing its elements without leaving any element out, that is, if there exists an involution without a fixed point. The property of a finite structure to be even is expressed by the IF sentence φ_{even}

$$\forall x \forall x'(\exists y/\{x'\})(\exists y'/\{x, y\})$$
$$[(x = x' \to y = y') \wedge (y = x' \to y' = x) \wedge y \neq x]$$

(For the details see Mann et al. (2011), example 4.15). It turns out that the sentence φ_{even} is true in a finite model (set) $\mathbb{M}$ if and only if there is an involution without a fixed point.

For a second example we consider infinity. A set $\mathbb{M}$ is (Dedekind) *infinite* if there is a function f which is an injection whose range is not the entire universe. This property is expressed by the IF sentences φ_{inf}

$$\exists w \forall x(\exists y/\{w\})(\exists z/\{w, x\})(x = z \wedge w \neq y)$$

whose truth conditions on a model $\mathbb{M}$ amount to the existence of two functions f and g and an individual c such that for all x we have $g(f(x)) = x$ and $f(x) \neq c$.

Now we observe that if Eloise and Abelard are single players (and not teams), in both examples Eloise lacks *action recall*, i.e., she does not rememeber her own moves. She also lacks *knowledge memory*: she forgets information she once knew (for instance, in the first game she knows the value of x at her first move but she forgets it at her second move.) The conjuction of action recall with knowledge memory is known in the literature as *perfect recall*. Thus in both examples greater expressive power is achieved through imperfect recall. A nice result by Sevenster (2006) (reproved in Mann et al. (2011), theorem 6.23) shows that if we want the games to be playable, that is, if we require the players to have perfect recall, then the expressive power of the relevant IF sentence does not exceed that of an ordinary first-order sentence.

3. SIGNALING GAMES

Van Benthem (2008) observes that evaluation games presuppose that the denotations of the basic lexical items such as predicates and object names have been settled. But there is still the legitimate question of how to account for the linguistic conventions that settle the meanings of these basic items. To this purpose *signaling games* have been developed from the 60's onward, stimulated by David Lewis' work on conventions. Lewis' work led to deeper connections between logic and game-theory explored by Parikh, Dekker & van Rooij, van Rooij, Jaeger & van Rooij. (See e.g., Pietarinen (2007)).

Lewis (1969) defines a *signaling problem* as a situation which involves a communicator (C) and an audience (A). C observes one of several states m which he tries to communicate or "signal" to A, who does not see m. After receiving the signal, A performs one of several alternative actions, called responses. Every situation m has a corresponding response $b(m)$ that C and A agree is the best response to take when m holds. Lewis argues that a word acquires its meaning in virtue of its role in the solution of various signaling problems.

To model a Lewisian coordination problem we fix the following elements:

- A set S of situations or states of affairs, a set Σ of signals, and a set R of responses.
- A function $b : S \to R$ which maps each situation to its best response.
- An encoding function $f : S \to \Sigma$ employed by C to choose a signal for every situation
- A decoding function $g : \Sigma \to R$ employed by A to decide which action to perform in response to the signal it receives.

A *signaling system* is a pair (f, g) of encoding and decoding functions such that $g \circ f = b$.

The standard example of a signaling problem is that of a driver who is trying to back into a parking space. She has an assistant who gets out of the car and stands in a location where she can simultaneously see how much space there is behind the car and be seen by the driver.

There are two states of affairs the assistant wishes to communicate, i.e., whether or not there is enough space behind the car for the driver to continue to back up. The assistant has two signals at her disposal: she can stand palms facing in or palms facing out. The driver has two possible responses: she can back up or she can stop.

There are two solutions to this signaling problem. The assistant can stand palms facing in when there is space, and palms facing out when there is no space, and *vice versa*. In the first case, the driver should continue backing up when she sees the assistant stand palms facing in, and stop backing up when the assistant stands palms facing out. In the second case, the driver should stop when he sees the assistant stands palms facing in, and back up when the assistant stands palms facing out. Both systems work equally well in the sense that the composition of the two communicating and responding strategies realize the best response: the driver backs up when there is space, and he stops when there is not.

We can model a Lewisian signaling system in IF logic by the sentence φ_{sig}

$$\forall x \exists z (\exists y / \{x\}) \{ (S(x) \to (\Sigma(z) \wedge R(y) \wedge y = x \}$$

and the model

$$\mathbb{M} = (M, S^M, \Sigma^M, R^M)$$

where

$$M = \{s_1, ..., s_n, t_1, ..., t_m\}$$
$$S^M = R^M = \{s_1, ..., s_n\}$$
$$\Sigma^M = \{t_1, ..., t_m\}.$$

The symbolism is self-explanatory: S stands for the set of states, Σ for the set of signals, and R for the set of responses. We prefered this simpler version where the task of the Audience is to identify the message ("$y = x$") to the original version where the task of the Audience is to perform the best action ("$y = b(x)$"). The modelization makes clear the "cooperative" nature of the signaling game. In the state x that he observes, the Communicator (C) represented by $\exists z$ sends a signal z to the Audience represented by $\exists z$. The latter tries to identify the state x from which z was sent. Although the semantical game $G(\mathbb{M}, \varphi_{sig})$ is strictly speaking a win-lose game of imperfect information, the logical form of the quantifier-free subformula of φ_{sig} makes it clear that

that existential team $\{\exists z, \exists y\}$ wins if the two players manage to "coordinate" in such a way that for every state x the respone y identifies x. In the case considered by Lewis in which the number m of signals equals the number n of states, there is a simple way for the two existential players to achieve successful coordination: The first existential player uses the signal t_i to signal the state s_i; and the second existential player decodes the signal t_i back into the state s_i. The pair of functions $h(s_i) = t_i$ and $k(t_i) = s_i$ (the other values do not matter) constitue a signaling system and encode the winning strategy of the existential team $\{\exists z, \exists y\}$. We notice how the property of IF logic to model the phenomenon of signaling, mentioned earlier, renders it adequate to express Lewisian signaling systems.

4. MODEL CONSTRUCTION GAMES

This is the famous tableau method cast in the form of a game between Builder and Critic. Unlike the case of evaluation games, we are not trying to evaluate a sentence in a model which interprets its logical vocabulary. Builder tries to build up a model where a given sentence would be true. Critic on the other side tries to show that such a model does not exist, and thereby the given sentence is a contradiction. These games are interesting, for they connect the notion of satisfiability and the notion of proof.

We fix a language L to which we add a countable set of C of constants to form the extended language L'. The tableaux we have in mind are the so-called *block* tableaux (Smullyan 1968, chapter XI) which are essentially the tableaux of Hintikka (1955). The points in the tableaux tree are finite sets of formulas and they become positions in the model construction games. There are two such sets that are finite: the Yes set (the set of true sentences) and the No set (the set of false sentences). At each stage of play of the game a position is reached. Critic selects a formula to be handled (either from the Yes or No set), after which Builder responds according to the rules listed below. In presenting the rules, I follow van Benthem (2008). However, my presentation of the winning and losing conditions differs from his.

We take some of the rules to be automatic:

a) If $\neg\varphi$ is in some box, then it changes to φ in the other.

b) If $\varphi \wedge \psi$ is in Yes, then it is replaced there by φ and ψ.

c) If $\varphi \vee \psi$ is in No, then it is replaced there by φ and ψ.

d) If $\exists x\varphi$ is in Yes, then it is replaced there by $\varphi(c)$, where c is a new constant which has not appeared earlier (in any box)

e) If $\forall x\varphi$ is in No, then it is replaced there with$\varphi(c)$, where c is a new constant which has not appeared earlier (in any box)

Here are the rules that govern the choices of the players:

f) Disjunction in Yes and conjunction in No prompt a move by Builder

g) If $\exists x\varphi$ occurs in No, then Critic chooses a constant c in the list of constants, and replaces $\exists x\varphi$ with $\varphi(c)$.

h) If $\forall x\varphi$ occurs in Yes, then Critic chooses a constant c in the list of constants, and replaces $\forall x\varphi$ with $\varphi(c)$.

In the block tableaux systems, what we can do at every stage depends only on the end points of the tree. The same holds in the corresponding games: what the players can do depends only on the plast position they have reached. We add one further constraint:

- Critic cannot select the same formula twice and he cannot select an atomic formula.

After a finite number of steps a play ends with a set of atomic formulas in the Yes box and a set of atomic formula in the No box. Here are the winning and loosing conventions for the play:

- The play is a win for Critic if an atomic formula occurs both in the Yes and the No box. Otherwise it is a win for Builder.

Clearly the games described here are extensive, win-lose games of perfect information. The notion of strategy for either one of the players and the notion of following a strategy remain standard.

Example Here is an example which illustrates the correlation between a winning strategy for Builder and the existence of a model (valuation) for the initial formula. The initial position is

$$\{[(s \vee r) \wedge (q \vee p)] \wedge \neg(q \vee (\neg p \vee r))\}; \emptyset.$$

Here is a winning strategy for Builder. After the automatic moves, the players reach the position

$$\{p, (s \vee r), (q \vee p)\} ; \{q, r\} .$$

If Critic schedules $(s \vee r)$, the players go to

$$\{p, (s \vee r)^*, (q \vee p)\} ; \{q, r\}$$

(the asterix indicates the formula scheduled by Critic) and if he schedules $(q \vee p)^*$, the players reach the position

$$\{p, (s \vee r), (q \vee p)^*\} ; \{q, r\} .$$

In the first case, let Builder choose s, and after Critic scheduling $(q \vee p)$, let her choose p. The play ends up with the position $\{p, s\} ; \{q, r\}$ which is a win for Builder. In the second case, let Builder choose p, and after Critic scheduling $(s \vee r)$, let her choose s. The play ends up with the same position as in the previous case which is a win for Builder. From this position we get a valuation for the initial sentence, by assigning True to the symbols in Yes and False to the symbols in No.

Example This game illustrates the correlation between a winning strategy for Critic in the model construction game with φ in the No box, and the logical validity of φ. Let φ be $\exists x(A \vee B) \to (\exists x A \vee \exists x B)$ that we rewrite as $\neg\exists x(A \vee B) \vee (\exists x A \vee \exists x B)$. We now take the root of the game-tree to be $\varnothing; \{\varphi\}$. The idea is that from the assumption that φ is false, a contradiction follows, hence a winning strategy for Critic. An automatic move leads to

$$\{A(c_1) \vee B(c_1)\} ; \{\exists x A, \exists x B\} .$$

Let Critic schedule the formula in the Yes box. Now, if Builder chooses $A(c_1)$, then let Critic schedule $\exists x A$ and then choose the constant c_1. Analogously, if Builder selects $B(c_1)$, then let Critic schedule $\exists x B$ and then choose the constant c_1. Notice Critic would not have a winning strategy if instead of scheduling $(A(c_1) \vee B(c_1))$ in the Yes box he would have schedulled any of the formulas $\exists x A, \exists x B$ in the No box. It is only the first strategy of Critic which compels Builder to reveal her winning strategy.

These two examples show nicely the difference in the strategies of the two players. Builder tries to maintain the consistency of the initial formulas whereas Critic's aim is to catch Builder in a contradiction. Hence Critic will force his opponent to reveal her strategy and then show it does not work.

4.1. Properties of model construction games

One can show (following methods analogous to the tableau methods) that the following are equivalent for first-order logic:

1. The set of formulas $\{\varphi_1, ..., \varphi_n, \neg\psi_1, ..., \neg\psi_m\}$ is satisfiable.

2. Builder has a winning strategy in the construction game with which starts with $\{\varphi_1, ..., \varphi_n\} ; \{\psi_1, ..., \psi_m\}$.

Alternatively, the following two claims are equivalent for forst-order logic:

3. The sentence $(\varphi_1 \wedge ... \wedge \varphi_n) \to (\psi_1 \vee ... \vee \psi_m)$ is a logical truth (i.e. it is provable).

4. Critic has a winning strategy in the construction game which starts with $\{\varphi_1, ..., \varphi_n\} ; \{\psi_1, ..., \psi_m\}$.

The direction from (1) to (2) gives an explicit correspondence between models and winning strategies for Builder. In fact if $\{\varphi_1, ..., \varphi_n, \neg\psi_1, ..., \neg\psi_m\}$ is satisfiable, starting from the root, in every immediate extension there is an open branch. That is, the moves for the conjunction and the universal quantifier do not close the branch. As for the disjunction and the existential quantifier, they preserve the satisfiability, so Builder is guaranteed to have a winning strategy.

The direction from (2) to (1) gives an explicit correspondence between winning strategy of Builder and models. Namely, if Builder has a winning strategy, then in all the histories in which Builder follows it, no contradiction appears. From the atomic formulas which appear in the Yes and No boxes (not all of them need to be taken into account), a Hintikka set is formed, from which a model for $\{\varphi_1, ..., \varphi_n, \neg\psi_1, ..., \neg\psi_n\}$ can be built by well known methods.

The equivalence between (3) and (4) is also straightforward. A winning strategy for the Critic allows him, for every possible move by

Builder, to reach a position which is a win for Critic. Notice, however, and this is one of the main differences with tableaus, that not all branches of the game tree are closed, but only those where the Critic's winning strategy is followed. Critic's winning strategies are explicitly correlated with proofs. To keep things simple, suppose $n = 0$ and $m = 1$. In the classical tableaus method one argues first that if ψ_1 is a first-order valid sentence, then there is a closed tableau starting with $\emptyset; \{\psi_1\}$. The tableau closes afer finitely many steps. An explanation of this is given using Köning's Lemma (each finitely branching infinite tree has an infinite branch): a closed infinite tableau is impossible because if the tableau is closed then every branch of it must be finite, hence the tableau must be finite. (See e.g. Smullyan 1968, p. 61.) In the present case Köning's Lemma is not needed: tableaux are finite, hence a winning strategy for Critic is a finite objects. The connection with proof comes quite naturally, and in the game-theoretical setting it goes back to the work in dialogical logic of Lorenzen & Lorenz (1978) (see also Rahman & Tulenheimo (2009)).

Now thinking about model construction games in extensive form, one has to keep in mind that from a winning strategy for Builder a Hintikka set (actually there may be more than one), and the corresponding model are formed from the set of histories in which the strategy is followed. And this set may be infinite, as in the following example.

Example We show that the sentence $\varphi_3 = \exists x \forall y \neg R(x, y)$ is not a logical consequence of the sentences $\varphi_1 = \forall x \forall y \forall z (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$ and $\varphi_2 = \forall x \neg R(x, x)$. The model construction game starts with $\{\varphi_1, \varphi_2\}; \{\varphi_3\}$. Here is a sketch of a winning strategy for Builder. If Critic schedules φ_3 then he has to choose a constant $c_i \in C$ for x after which the play moves automatically to $\{\varphi_1, \varphi_2, R(c_i, c_{i+1})\}; \emptyset$. Critic has now to schedule either φ_1 or φ_2. If the latter, Critic has to choose a constant $c_m \in C$ and the play moves to $\{\varphi_1, R(c_i, c_{i+1})\}; \{R(c_m, c_m)\}$. Let us simplify things, and suppose Critic chooses three constants, c_n, c_p and c_r in C as values for x, y and z. This takes the play to

$$\left\{\neg(R(c_m, c_p) \wedge R(c_p, c_r)) \vee R(c_m, c_r)), R(c_i, c_{i+1})\right\}; \{R(c_m, c_m)\}$$

We show that this is a winning position for Builder. There are several cases. Case 1. $m = p$. Let Builder choose the left disjunct,

which will take the game to

$$\{R(c_i, c_{i+1})\}; \left\{R(c_m, c_m), (R(c_m, c_p) \wedge R(c_p, c_r))\right\}$$

and then let her choose left, ending the play with

$$\{R(c_i, c_{i+1})\}; \{R(c_m, c_m)\}$$

The play is a win for Builder. Case 2. $m \neq p$. We have two subcases. Subcase 21. $m = i$. Let Builder choose left, ending in the same position as in the previous case, and then choose right. The play ends with

$$\{R(c_i, c_{i+1})\}; \left\{R(c_m, c_m), R(c_p, c_r)\right\}$$

The play is a win for Builder, because $i \neq p$. Subcase 22. $m \neq i$. We have two subcases. Subcase 221. $i = p$. Let Builder choose left, and then left again, ending the play with

$$\{R(c_i, c_{i+1})\}; \left\{R(c_m, c_m), R(c_m, c_p)\right\}.$$

The branch is open, so it is a win for Builder. Subcase 222. $i \neq p$. Let Builder choose left and then right. The play ends with

$$\{R(c_i, c_{i+1})\}; \left\{R(c_m, c_m), R(c_p, c_r)\right\}$$

This branch is also open. The other cases (when Critic schedules the other formulas) are dealt with in a similar way. We now form a Hintikka set H from the set of histories where the above strategy is used. Notice that this strategy is followed in all the branches labelled by

$$\{\varphi_1, R(c_i, c_{i+1})\}; \{R(c_m, c_m)\}$$

Let $R(c_i, c_{i+1}) \in H$ for every natural number i, and $\neg R(c_m, c_m) \in H$, for every natural number m. It remains to show that $R(c_i, c_{i+2})$ appears in the branches where the strategy is used, for every i. The winning strategy has to win against any scheduling choices of Critic, also against him scheduling φ_2 before φ_3 (and φ_1). In that case Critic will choose a constant c_{i+1} and the play will reach

$$\{\varphi_1, \neg R(c_{i+1}, c_{i+1})\}; \{\varphi_3\}$$

An automatic move will take the play to

$$\{\varphi_1\}\,;\{\varphi_3, R(c_{i+1}, c_{i+1})\}\,.$$

Now let Critic schedule φ_3. Then he will have to choose a constant $c_i \in C$ which will take the play to

$$\{\varphi_1\}\,;\{\forall y \neg R(c_i, y), R(c_{i+1}, c_{i+1})\}$$

and then, after an automatic move to

$$\{\varphi_1, R(c_i, c_{i+2})\}\,;\{R(c_{i+1}, c_{i+1})\}$$

Given that Builder did not make any move, his winning strategy is also trivially followed in all these branches. Thus $R(c_i, c_{i+2}) \in H$.

We can relax the game and give the players more freedom on some of the automatic moves. For instance, let us replace (d) and (e) with (d*) and (e*) respectively:

d*) If $\exists x\varphi$ is in Yes, then let Builder choose a constant $c \in C$ and replace $\exists x\varphi$ by $\varphi(c)$.

e*) If $\forall x\varphi$ is in No, then let Builder choose a constant $c \in C$ and replace $\forall x\varphi$ by $\varphi(c)$.

The relaxation makes the interaction of the players more dynamic. Consider the following example (used by Coquand in a different context) where a game is played with the logically valid formula φ :

$$\exists x \forall y R(x, y) \vee \forall z \exists w \neg R(z, w).$$

Here is a winning strategy for Critic in the model construction game which starts with $\varnothing; \{\varphi\}$. After the automatic move which leads to

$$\varnothing; \{\exists x \forall y R(x, y), \forall z \exists w \neg R(z, w)\}$$

let Critic schedule $\forall z \exists w \neg R(z, w)$, after which Builder has to choose a constant $c \in D$:

$$\varnothing; \{\exists x \forall y R(x, y), \exists w \neg R(c, w)\}\,.$$

Let Critic schedule $\exists x \forall y R(x, y)$ and then choose the same constant c for x. This will take the play to

$$\varnothing; \{\forall y R(c, y), \exists w \neg R(c, w)\}\,.$$

Let Critic now schedule $\forall y R(c, y)$. By (e*) Builder has to choose a constant $d \in C$ $(d \neq c)$ after which the players move to

$$\varnothing; \{R(c, d), \exists w \neg R(c, w)\}\,.$$

Finally let Critic choose d for w. This strategy is winning. Notice once again, how, by an appropriate scheduling of the formulas, Critic forced his opponent to reveal her strategy.

5. PLAYABILITY OF GAMES: RECURSIVE STRATEGIES

There is a major difference between the two kinds of games analyzed so far: in evaluation games the strategies of Eloiser correlate with the (material) truth of a given sentence φ, whereas in the model construction games the strategies of Critic correlate with proofs of φ. Both are grounds for asserting φ but the difference lies in their "epistemic accessibility". A proof in a first-order formal system is an effective notion, whereas truth is not. Hintikka (1996) makes a proposal to overcome the difference: to make semantical games playable by restricting the strategies of the two players to effective (recursive) ones:

> The demand of playability might seem to imply that the set of the initial verifier's strategies must be restricted. For it does not seem to make any sense to think of any actual player as following a nonconstructive (nonrecursive) strategy. How can I possibly follow in practice such a strategy when there is no effective way for me to find out (or perhaps even know) in general what my next move will be? [...]For the basis of my argument was the requirement that the semantical games that are the foundations of our semantics and logic must be playable by actual human beings, at least in principle. This playability of our "language games" is one of the most characteristic features of the thought of both Wittgenstein and Dummett. (Hintikka 1996, pp. 214–215)

In Boyer & Sandu (2012) this proposal was amended in two ways. The requirement of the recursivity of strategy functions is well defined only on the universe of natural numbers. But even then, this requirement would be better served if we supplemented this restriction with another one: that the atomic formulas which mark the end of semantical games be decidable. All in all we propose to restrict semantical games to be played only on recursive structures, i.e. structures where the relations and function symbols of the arithmetical language are interpreted by recursive relations and functions. Indeed, if our model is not recursive, speaking of playability does not make much sense, since in that case, even the truth of simple atomic formulas cannot be computed. (In fact, the decidability of the atomic formulas was also an essential ingredient in Dummett's notion of truth.) So let us amend Hintikka's initial proposal and reduce truth to what we shall call "CGTS-truth" (computable game-theoretical semantics truth):

- A first-order sentence φ is $CGTS$-true on a recursive model $\mathbb{M}$, $M \models_{CGTS} \varphi$, exactly when there is a computable winning strategy for Eloise in the semantical game played with φ on $\mathbb{M}$. (When φ is a formula, this definition is appropriately relativized to an assignment.)

We shall consider also formal systems for arithmetic, like Peano Arithmetic, *PA*. Since the standard model $\mathbb{N}$ of arithmetic is the only recursive structure of $\mathbb{PA}$ (up to isomorphism), we consider only effective winning strategies for Eloise in semantical games played on $\mathbb{N}$.

We now hope to obtain a match, at least a partial one, between two effective notions: $CGTS$-truth on one side and proof in formal systems for arithmetic, say Peano Arithmetic (*PA*), on the other. In other words, following Bonnay (2004) and Boyer & Sandu (2012), we ask the following questions:

(1) Do proofs in *PA* yield $CGTS$-truth? That is, for an arbitrary formula φ in elementary arithmetic, do we have

$$PA \vdash \varphi \Rightarrow PA \models_{CGTS} \varphi?$$

where $\Gamma \models \varphi$ is defined by: for all recursive models , if for all $\psi \in \Gamma$, $\mathbb{M} \models_{CGTS} \psi$, then $\mathbb{M} \models_{CGTS} \varphi$.

(2) Can the $CGTS-$ truth of an arithmetical senntence be always interpreted as given by a proof? That is, is there a sound and effective proof procedure $\vdash_{CGTS}$ that is complete for arithmetical $CGTS-$ truth

$$PA \models_{CGTS} \varphi \Longleftrightarrow PA \vdash_{CGTS} \varphi?$$

It is well known that the answers to both questions are negative. This shows that Hintikka's ameliorated proposal to to restrict the winning strategies to computable ones on recursive models will not help to obtain a positive result to any of the two questions. In Boyer & Sandu (2012) we looked to new games in order to obtain a positive result to (1). Let us say few words about this.

5.1. Games with backward moves

Coquand (1995); Krivine (2003); Bonnay (2004) define new games in order to obtain a positive result to (1). Such a result cannot be obtained as long as strategies in semantical games are formulated as we did in the first section. The reason for this is simply that there are not enough of them to correspond to proofs. To enlarge the set of of verifying winning strategies, standard semantical games need to be modified. One way to do so is to allow for backward moves. Another way is to enrich the set of classical connectives, as in Japaridze's Computational logic (see, e.g. Japaridze (2003)). We focus here on the former. The main differences between standard semantical games and games with backward moves is that in the latter, whenever it is her turn to move, Eloise can return to any one of her ealier decision points and remake her choice, or whenever the players reach a false atomic formula, Eloise can prolong the game by returning to any of her ealier decision points, and remake her choice (in both cases the play continues as in the stadard game.) The winning and loosing conditions are now:

(c) Eloise wins a play if it is finite and it ends up with a true atomic formula; otherwise Abelard wins.

Overlooking some important details, we give an example.

Example We consider the game played with

$$\exists m \forall x (x \leq m) \vee \forall m \exists x (x > m)$$

on the universe of natural numbers $\mathbb{N}$ with their standard ordering. Here is a winning strategy for Eloise:

(1) Eloise chooses the right disjunct

(2) Abelard chooses $m_0 \in \mathbb{N}$ as a value for m

(3) Eloise changes her mind and goes back to an earlier decision point: she now chooses the left disjunct.

(4) Eloise chooses m_0 as a value for m

(5) Abelard chooses a value x_0 as a value for x

(6) If $x_0 \leq m_0$ then Eloise wins.

(7) If $m_0 < x_0$, then Eloise goes back to her initial position and where she chose the left disjunct and decides now to continue the game in the right disjunct choosing the value x_0 as a value for x.

In model construction games we mentioned the connection between Critic's winning strategies with proofs. By the appropriate scheduling of tasks, Critic forces Builder to reveal her strategy and then catches her in a contradiction. In the present case the same idea reappears in connection with Eloise's strategy: she forces Abelard to disclose his strategy, and then, as the example shows, she wins independently of whether the atomic formula $x \leq m$ is true or false in the underlying model. In both cases the winning strategies of Critic and Eloise are independent of the underlying model. This is not the case with Hintikka style evaluation games (GTS), where the winning strategy of Eloise always depends on the underlying model, even in games like $G(\mathbb{M}, \varphi \vee \neg\varphi)$ where $\varphi \vee \neg\varphi$ is a logical truth. We have here a distinction between "for every model $\mathbb{M}$ there is a winning strategy..." and "there is a winning strategy such that for every model..." (Japaridze 2003).

Bonnay (2004) gives the following result:

If $\mathbb{M}$ is a recursive model, π is a proof of $\Gamma \vdash \varphi$ and recursively winning strategies $\{f_i\}_{\varphi_i}$ for Eloise are given for each game $G(\mathbb{M}, \varphi_i)$ with backwards moves, $\varphi_i \in \Gamma$, then π recursively yields a recursively winning strategy for Eloise in the game with backwards move $G(\mathbb{M}, \varphi)$.

This result establishes the connection between proofs and semantical games we have been looking for. In the particular case in which $\Gamma = PA$, a proof π of the sentence φ yields an effective winning strategy for Eloise in the game $G(\mathbb{M}, \varphi)$. In the particular case in which $\Gamma = \emptyset$, a proof of φ yields an effective strategy for Eloise in the game with backward moves $G(\mathbb{M}, \varphi)$.

And now few historical notes. Coquand (1995) has introduced games with backward moves and reformulated Gentzen's and Novikoff's "finitist" sense of an arithmetical proposition as a winning strategy for the game associated with it. He also proved that classical proofs are admissible in the sense that starting with winning strategies for premisses a proof would effectively yield a winning strategy for the conclusion. There is a similar link between classical proofs and winning strategies in games with backwards moves in Hayashi (2007) and in Tait (2005). Bonnay (2004) used Coquand's games with backward moves to give a "constructibly acceptable" version of Hintikka's GTS. In this context Krivine (2003 pp. 272–274) proved another important result, mentioned also in Bonnay (2004) and hinted at in Tait (2005): in the setting of games with backward moves, Eloise has a winning strategy (on the universe of natural numbers) if and only if she has a computable winning strategy. Thus we see how allowing for more strategies in the games, we get, as a bonus, also computable strategies which are more in line with "playability of games". The playability of games here is implemented at the level of strategies, and not at the level of information sets as in our earlier discussion of perfect recall.

6. RANDOMIZING STRATEGIES: STRATEGIC IF GAMES

The procedure of combining mixed strategies with IF games has been suggested (in the context of the branching quantifiers) by Blass & Gurevich (1986) following a suggestion given by Ajtai. This procedure has been studied in details for the first time in Sevenster (2006) and developed in Sevenster & Sandu (2010). What is distinctive about this procedure by comparison with semantical games and model construction games is that the starting points are no longer logical notions characterized in game-theoretical terms, but rather game-theoretical concepts like Nash equilibrium that we try to capture in some logical language.

We return to IF logic.

For φ and IF sentence and $\mathbb{M}$ a finite model, we convert a semantical IF game $G(\mathbb{M},\varphi)$ into a strategic IF game $\Gamma(\mathbb{M},\varphi) = (S_\exists, S_\forall, u_\exists, u_\forall)$ where

- $S_\exists$ is the set of all possible strategies of Eloise in $G(\mathbb{M},\varphi)$
- $S_\forall$ is the set of all possible strategies of Abelard in $G(\mathbb{M},\varphi)$
- $u_\exists(\sigma,\tau)$, the payoff of Eloise when she playes the strategy σ and Abelard plays the strategy τ, is computed in the following way. The strategies σ and τ generate a maximal history h in the extensive game $G(\mathbb{M},\varphi)$. If that history is a win for Eloise in $G(\mathbb{M},\varphi)$, let $u_\exists(\sigma,\tau)$ be 1; otherwise let it be 0.
- $u_\forall(\sigma,\tau)$ is defined analogously.

Obviously $\Gamma(\mathbb{M},\varphi)$ is a finite, win-lose strategic game.

Let us fix a strategic IF game $\Gamma(\mathbb{M},\varphi) = (S_\exists, S_\forall, u_\exists, u_\forall)$. A mixed strategy ν for player $p \in \{\exists,\forall\}$ is a probability distribution over S_p, that is, a function $\nu : S_p \to [0,1]$ such that $\sum_{\tau\in S_i}\nu(\tau) = 1$. ν is uniform over $S_i{}' \subseteq S_i$ if it assigns equal probability to all strategies in $S_i{}'$ and zero probability to all the strategies in $S_i - S_i{}'$. Given a mixed strategy μ for player $\exists$ and a mixed strategy ν for player $\forall$, the expected utility for player p is given by:

$$U_p(\mu,\nu) = \sum_{\sigma\in S_\exists}\sum_{\tau\in S_\forall}\mu(\sigma)\nu(\tau)u_p(\sigma,\tau).$$

We can simulate a pure strategy σ with a mixed strategy which assigns to σ probability 1. That is, when $\sigma \in S_\exists$ and ν is a mixed strategy for player $\forall$, we let

$$U_p(\sigma,\nu) \quad = \quad \sum_{\tau\in S_\forall}\nu(\tau)u_p(\sigma,\tau)$$

Similarly if $\tau \in S_\forall$ and μ is a mixed strategy for player $\exists$, we let

$$U_p(\mu,\tau) \quad = \quad \sum_{\sigma\in S_\exists}\mu(\sigma)u_p(\sigma,\tau).$$

Example Let us look at the strategic IF games $\Gamma(\mathbb{M},\varphi_{MP})$ and $\Gamma(\mathbb{M},\varphi_{IMP})$, where $\mathbb{M} = \{1,2,3,4\}$, $\varphi_{MP} = \forall x(\exists y/\{x\})x = y$ and $\varphi_{IMP} = \forall x(\exists y/\{x\})x \neq y$:

	1	2	3	4
1	(1,0)	(0,1)	(0,1)	(0,1)
2	(0,1)	(1,0)	(0,1)	(0,1)
3	(0,1)	(1,0)	(1,0)	(0,1)
4	(0,1)	(0,1)	(0,1)	(1,0)

	1	2	3	4
1	(0,1)	(1,0)	(1,0)	(1,0)
2	(1,0)	(0,1)	(1,0)	(1,0)
3	(1,0)	(1,0)	(0,1)	(1,0)
4	(1,0)	(1,0)	(1,0)	(0,1)

6.1. *Nash equilibrium semantics: some properties*

Let $\Gamma(\mathbb{M},\varphi) = (S_\exists, S_\forall, u_\exists, u_\forall)$ be a strategic IF game. The pair Let (μ,ν) is an equilibrium if the following two conditions hold:

- for every mixed strategy μ', of Eloise, $U_\exists(\mu,\nu) \geq U_\exists(\mu',\nu)$
- for every mixed strategy ν', of Abelard, $U_\forall(\mu,\nu) \geq U_\forall(\mu,\nu')$.

A well known result in classical game theory, von Neumann's Minimax Theorem (Neumann 1928) tells us that

- every finite, constant sum, two-player game has an equilibrium in mixed strategies
- every two such equilibria have the same expected utility.

These two results garantee that we can talk about *the value of a strategic IF game*: it is the expected utility returned to player $\exists$ by any equilibrium in the relevant strategic game. We can now define *the value of the IF sentence* φ in the finite model $\mathbb{M}$: it is the value of the strategic game $\Gamma(\mathbb{M},\varphi)$.

Mann et al. (2011) shows that this interpretation is a conservative extension of the old game-theoretical semantics, in the following sense:

(i) $\mathbb{M} \models_{GTS}{}^{+}\varphi$ iff the value of φ on $\mathbb{M}$ is 1

(ii) $\mathbb{M} \models_{GTS}{}^{-}\psi$ iff the value of φ on $\mathbb{M}$ is 0.

Let us return to our earlier examples $\Gamma(\mathbb{M},\varphi_{MP})$ and $\Gamma(\mathbb{M},\varphi_{IMP})$, where $\mathbb{M} = \{1,2,3,4\}$, $\varphi_{MP} = \forall x(\exists y/\{x\})x = y$ and $\varphi_{IMP} = \forall x(\exists y/\{x\})x \neq y$. Let μ and ν be uniform probability distributions over $\{1,...,4\}$. We may check that the pair (μ,ν) is an equilibrium in both games and that the value of φ_{MP} on $\mathbb{M}$ is $1/4$ and that of φ_{IMP} is $3/4$.

We extend the object language to include identities of the form $NE(\varphi) = r$. The corresponding semantic clause is:

- $\mathbb{M} \models NE(\varphi) = r$ if and only if the value of φ in $\mathbb{M}$ is r.

It follows from results in Mann et al. (2011), Chapter 7, that the Nash equilibrium semantics satisfies the following principles of *Weak Lukasiewicz logic:*

P1 $\quad NE(\varphi \vee \psi) = max(NE(\varphi), NE(\psi))$

P2 $\quad NE(\varphi \wedge \psi) = min(NE(\varphi), NE(\psi))$

P3 $\quad NE(\neg\varphi) = 1 - NE(\varphi)$.

From these principle we can weakly derive Kolmogorov axioms of probabilities:

Ax1 $\quad NE(\varphi) \geq 0$

Ax2 $\quad NE(\varphi) + NE(\neg\varphi) = 1$

Ax3 $\quad NE(\varphi) + NE(\psi) \geq NE(\varphi \vee \psi)$

Ax4 $\quad NE(\varphi \wedge \psi) = 0 \rightarrow NE(\varphi) + NE(\psi) = NE(\varphi \vee \psi)$.

It is well known that there are some other conceptions of probability that verify Kolmogorov's axioms: probabilities as statistical frequencies, and probabilities as degrees of belief. The game-theoretical approach seems to offer a new interpretation of Kolmogorov's axioms: as probabilities arising from the Nash equilibrium semantics. It introduces new structure on the class of semantically indeterminate sentences for a given model. The question of the extent to which this is really a genuinely new interpretation of the axioms and the question of its relationships to the other two interpretations are matters for future research.

Notes

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