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THE 3ⁿ CIRCULAR FACTORIAL DESIGNS: MORE ROBUST ESTIMATE OF ALL FACTORIAL EFFECTS

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ABSTRACT

A 3ⁿ circular factorial design which facilitates estimates of all factorial effects providing internal partial confounding with blocks within each of its diameters in different directions is constructed for use in analyzing the multi-directional data of different disciplines. The design possesses small error variance by eliminating the heterogeneity in different directions. An example is given.

Keywords: Factorial Design, Multidirectional Heterogeneity, Circular Factorial and Diameter

1. INTRODUCTION

The general factorial designs in blocks [Bose and Kishen (1940), Bose (1947) and Kempthorne (1947)] provides the analysis of one-directional data eliminating the heterogeneity in one direction and losing information on some confounded effects. Kempthorne (1947), Rao (1946), Cochran and Cox (1957) and others discussed the latin square or quasi-latin square forms of different factorial designs. Patterson (1976), Bailey (1977) and Patterson and Bailey (1978) set up various row – column factorial designs with one unit per cell. John (1987) also discussed the factorial structures of row-column designs. Shamsuddin (1987, 1989, 1991, 1993, 1994), Aggarwal and Shamsuddin (1987) and Shamsuddin and Aggarwal (1988, 1989) constructed square/ rectangular confounded different full and fractional factorial designs with multiple/single unit per cell eliminating heterogeneity in row and column directions. Those designs arranged in row-column structures can be used to analyze the data in row and column direction eliminating the heterogeneity in both directions by losing the information of some confounded effects. Shamsuddin (1994, 1997) discussed the circular factorial designs which controls multi-directional heterogeneity and helps to estimate all factorial effects providing internal partial confounding. We discuss how any circular factorial design provides smaller variance (proof is shown by a theorem) and helps to analyze the multi-directional data with the estimation of all factorial effects controlling the heterogeneity in different directions. Since in different experimental fields of different sectors the internal contents of the experimental units are almost always unknown, might be different and complex, their movements or directions of variability and variability contour within and between the experimental units are also unknown, and might

be also different and complex. For example, in an agricultural experimental field the direction of soil fertility and soil heterogeneity between experimental units are unknown, might be different and complex ; it might move in all possible directions within the experimental field. So one or two directional experimental designs may not always be well suited for agricultural experimentation. In such cases, the circular factorial designs are better to use for controlling multi-directional heterogeneity and data obtained from different directional diameters in such designs can be termed as multi-directional data. The analysis of data obtained through these designs will provide better inference about the parameters of the model concerned. In this matter, we shall discuss here the particular case of 3^n circular factorial design showing in brief its construction pattern for using and analyzing a multi-directional data set with an example.

Theorem 1. *The circular factorial design provides small error variance and precise estimates of factorial effects.*

Proof. In each model we have the observed value $Y = \text{true value } Z\beta + \text{random error } \varepsilon$ i.e. $Y = Z\beta + \varepsilon$, where ε_i is distributed as $N(0, \sigma^2)$. In all designs except the completely randomized design there exist different types of errors of heterogeneity in experimental fields. In the randomized block design we control one-directional error of heterogeneity through blocking and in the latin square / row-column design we control the effects due to two-directional errors of heterogeneity as row and column effects. In the circular factorial design we control multi-directional errors of heterogeneity. Besides these effects of heterogeneity the usual error ε of randomness in each heterogeneous experimental field exists. If a design is not capable of controlling these effects of heterogeneity, its existing heterogeneity will disturb estimators of model parameters. We shall observe the model and the estimation technique of its parameters as follows:

We have the model $Y = Z\beta + \varepsilon$ when there is only the random error, but there is no error of heterogeneity or when the error of heterogeneity is fully controlled. But when there exists error of heterogeneity or when the error of heterogeneity is not fully controlled the model becomes either $Y^* = Z\beta + \varepsilon^*$. . . (i) assuming the true value $Z\beta$ unchanged and inflating the error ε to ε^* or as $Y^* = Z\beta^* + \varepsilon$. . . (ii) assuming the random error ε unchanged and changing the true value $Z\beta$ to other value $Z\beta^*$, where $Y^* = Y + \delta$, $\delta = \text{effect of heterogeneity}$, $Z\beta^* = Z\beta + \delta$ and $\varepsilon^* = \delta + \varepsilon$. So, $\text{var}(\varepsilon^*) = \text{var}(\varepsilon) + \text{var}(\delta) + 2\text{cov}(\varepsilon, \delta)$, or $\text{var}(\varepsilon^*) > \text{var}(\varepsilon)$. . (iii) That is, every design provides a small error variance $\text{var}(\varepsilon)$ when the effect of heterogeneity is fully controlled and a greater error variance $\text{var}(\varepsilon^*)$ when the effect of heterogeneity is not fully controlled. In this situation when there exist any heterogeneity the estimate of $\beta = (Z'Z)^{-1}Z'Y^* = (Z'Z)^{-1}Z'(Z\beta + \varepsilon^*) = (Z'Z)^{-1}Z'Z\beta + (Z'Z)^{-1}Z'Z\varepsilon^*$, and $E(\beta) = \beta$ if and only if $E(\varepsilon^*) = 0$. But $E(\varepsilon^*) = E(\delta) + E(\varepsilon) = E(\delta) \neq 0$ i.e. β can not be estimated by the simple least square procedure directly and hence shows a wrong estimate . . . (iv). In (ii), the estimate $\beta^* = (Z'Z)^{-1}Z'Y^* = (Z'Z)^{-1}Z'(Z\beta + \delta + \varepsilon) = (Z'Z)^{-1}Z'Z\beta + (Z'Z)^{-1}Z'\delta + (Z'Z)^{-1}Z'\varepsilon$. . . (v) $= \beta + (Z'Z)^{-1}Z'\delta + (Z'Z)^{-1}Z'\varepsilon = \beta + \text{non zero quantity}$, i.e. $E(\beta^*) \neq \beta$, so β^* is a biased estimate of β . Therefore, unless we exclude this existing δ error/effect of heterogeneity from our estimation process we cannot obtain the unbiased estimate .

2. 3^n Circular Factorial Design

The circular 3^n factorial design (Shamsuddin 1994, 1997) is the mathematical arrangement of the treatments of a 3^n factorial in $(3^k-1)/(3-1)$ diameters where $0 < k \leq n$. Each diameter possesses 3 blocks including centre block and intersects with all other diameter at the centre block i.e. the central block is common to all diameters. So the comparison between the blocks including the central block within a diameter will determine the block-effect of the data in the direction of the diameter providing the internal partial confounding with blocks within the diameter by controlling heterogeneity in that direction.

To construct a 3^n circular factorial design with $(3^k-1)/(3-1)$ diameters with $3^{[n-(k-1)]}$ treatment units in each diameter of 3 blocks, each having $3^{(n-k)}$ treatments ($n \geq k$), the design contains $(3^n-1)/(3-1)$ factorial effects out of which $(3^k-1)/(3-1)$ effects in total are confounded with blocks of $(3^k-1)/(3-1)$ diameters respectively, so that each diameter contains only one internally partially confounded factorial effect. Out of $(3^k-1)/(3-1)$ confounded factorial effects, k effects are independently confounded within any k diameters respectively (Shamsuddin 1994), as for example, if the factorial effect V_1 is internally, partially, and independently confounded with the first diameter and the effect V_2 independently, partially and internally confounded with the second diameter, then the generalized factorial effect V_1V_2 and $V_1V_2^2$ are respectively confounded internally partially with the third and fourth diameters. These two independent factorial effects V_1 and V_2 and generalized factorial effects V_1V_2 and $V_1V_2^2$ are respectively represented by the following sets of equations, in other words, the four diameters partially confounding respectively V_1 , V_2 , V_1V_2 and $V_1V_2^2$ with blocks within themselves are given by the following (3.1-3.4) equation sets

$$\begin{array}{ll} a_1x = 1, 0, 2 & a_1x = 0, 0, 0 \\ a_2x = 0, 0, 0 & a_2x = 1, 0, 2 \end{array} \quad (3,1) \quad (3,2)$$

$$\begin{array}{ll} a_1x = 1, 0, 2 & a_1x = 1, 0, 2 \\ a_2x = 1, 0, 2 & a_2x = 2, 0, 1 \end{array} \quad (3,3) \quad (3,4)$$

$$\text{where } a_i = [a_{i1} \ a_{i2} \ \dots \ a_{in}]; \ x = (x_1 \ x_2 \ \dots \ x_n)'; \ 0,1,2, a_{ij}, x_i \in \text{GF}(3).$$

and, in general, for constructing a 3^n circular factorial design of $(3^k-1)/(3-1)$ diameters, a diameter internally confounds partially an effect $V_1^{\alpha_1} V_2^{\alpha_2} \dots V_k^{\alpha_k}$ which is generally represented by the sets of equations

$$\begin{array}{ll} a_1x = & \alpha_1, 0, 2\alpha_1 \\ a_2x = & \alpha_2, 0, 2\alpha_2 \\ \dots & \dots \\ a_kx = & \alpha_k, 0, 2\alpha_k, \end{array}$$

$$\text{where } a_i = (a_{i1} \ a_{i2} \ \dots \ a_{in}), \ x = [x_1 \ x_2 \ \dots \ x_n]'; \ a_{ip}, x_i, u_{ij}, \alpha_i \in \text{GF}(3)$$

In illustrating this procedure let us consider the case of 3^3 circular factorial designs in four directional diameters of 3 blocks with multiple units.

2.1 3^3 circular factorial design with 3 units per block

This design with 3 treatments per block can be obtained by internally confounding partially two independent factorial effects (here $k = 2$) and in all $(3^2-1) / (3-1) = 4$ factorial effects. Let the first and the second diameter internally confound partially the effects AB and AC (here $V_1 = AB$ and $V_2 = AC$) respectively. By the equations (3.1- 3.4 or 3.5 –3.8) we get 4 diameter of the design (Table 2.1)

$$\begin{array}{llll} x_1+x_2 = 1,0,2 & (3.5); & x_1+x_2 = 0,0,0 & (3.6); & x_1+x_2 = 1,0,2 & (3.7); & x_1+x_2 = 1,0,2 & (3.8) \\ x_1+x_3 = 0,0,0; & & x_1+x_3 = 1,0,2; & & x_1+x_3 = 1,0,2; & & x_1+x_3 = 2,0,1 \end{array}$$

Table 2.1 The 3^3 circular design with 3 treatments per block.

	001	
	212	
	120	
100		101
222		220
011		012
010	000	020
102	122	112
221	211	201
110		200
202		022
021		111
	002	
	121	
	210	

In this circular factorial design having 3 treatments per block the effect AB is internally partially confounded with 3 blocks in the first diameter, the effect AC is internally partially confounded in the second, AB^2C^2 in the third and BC^2 is internally partially confounded in the fourth diameter. But none of the factorial effects is completely confounded in full factorial.

The effect confounded with 3 blocks of a diameter will lose the information equal to the block-effect of that diameter. This lost information is due to the effect of heterogeneity in the direction of that diameter. The amount of heterogeneity in the direction of a diameter in which a factorial effect is internally partially confounded will be deducted from the total estimated value (obtained

from full factorial by existing procedure) of the factorial effect to get the actual estimate of that factorial effect. In these circular factorial designs no effect loses any degree of freedom and the model for the above 3^3 factorial circular design is given by

$$Y_{ijk(lmn)r} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \varepsilon_{ijk(r)},$$

where, $i=1,2,3; j=1,2,3; k=1,2,3; r=2$.

The estimates of all factorial effects are obtained as usual procedure for normal full factorials except the internally partially confounded effects AB, AC, BC^2 and AB^2C^2 . The estimate of each internally partially confounded effect will be usual estimate minus the block effects in the diameter in which it is internally partially confounded and thus the sums of squares of such confounded factorial effects are obtained. The block effect in any diameter does not have any degree of freedom, since any partially confounded effect which possesses same d.f. as like as unconfounded effect by existing procedure does not lose any d.f. Thus the estimates of all factorial effects are more robust in comparison to those in this type of factorial 3^n designs.

The sums of squares of 4 diameters in 4 directions in the circular factorial in the following way:
 $SS(\text{ith Diameter}) = ss(\text{Blocks in ith direction in ith diameter}) = ss(\text{effect due to heterogeneity in ith direction}) = [B^2_{i1} + B^2_{i(0)} + B^2_{i2}]/3r - C.T., i=1,2,3,4$

3. DATA AND RESULT

Let us decide to use a 3^3 circular factorial of 3^3 treatment combinations having 2 replications of 3 units per block for the experimentation in a heterogeneous field (like hilly field) containing heterogeneous plot of land. Let the experiment of a 3^3 circular factorial design of three fertilizer factors A, B, C be conducted on such a land of agriculture and following data be obtained from the layout of the experiment of the circular factorial design (Table 2.1):

000(5.2, 5.8), 010(13.2, 15.5), 020(15.8, 16.2), 001(11.5, 12.5), 100(18.3, 17.9), 021(17.6, 17.5), 002(15.8, 16.7), 012(18.2, 18.6), 200(25.7, 24.5), 211(29.9, 30.5), 221(34.5, 35.8), 201(26.3, 26.4), 212(32.4, 33.6), 011(16.8, 17.2), 202(34.5, 33.9), 210(27.4, 29.5), 220(30.2, 31.2), 111(20.8, 22.7), 122(27.9, 28.2), 102(25.7, 24.7), 112(24.6, 25.9), 120(21.2, 20.8), 222(33.8, 36.8), 110(20.2, 20.8), 121(25.5, 26.2), 101(21.6, 21.8), 022(19.7, 18.9).

The estimate of the total effect of unconfounded effects are obtained by usual procedure, but the estimates (obtained by usual procedure) of the total effects of the internally and partially confounded AB, AC, BC^2 , AB^2C^2 , are to be adjusted. Thus the least square estimate of total effect of AB based on total number of observation is $(AB)_0 = 400.6$, $(AB)_1 = 431.7$, $(AB)_2 = 412.0$ and by subtracting the block-effect of the diameter (in which AB is internally confounded) from its least square estimate its actual estimate of total effect at the level 0 is $(AB)_0 = 400.6 - [000(11.0) + 122(56.1) + 211(60.4)] = 400.6 - 127.5 = 273.1$, $(AB)_1 = 431.7 - [010(28.7) + 102(50.4) + 221(70.3)] = 431.8 - 149.4 = 282.4$, $(AB)_2 = 412.3 - [201(52.7) + 112(50.5) + 020(32.0)] = 412.3 - 135.2 = 277.1$ and hence the actual $ss(AB) = 3.63$. Similarly by subtracting the block-effects from the least square estimates we get the $ss(AC) = 6.75$, $SS(BC^2) = 47.0$ and $ss(AB^2C^2) = 4.00$. The estimates of other factorial as same as least square estimates and thus we find that $ss(\text{total}) = 2922.1$, $ss(\text{rep}) = 5.16$, $ss(A^*) = 1802.56$, $ss(B^*) = 108.35$, $ss(AB^2) = 7.65$, $ss(C^*) = 198.42$, $ss(AC^2) = 35.39$, $ss(BC) = 15.92$, $ss(ABC) = 2.11$, $ss(ABC^2) = 32.49$, $ss(AB^2C) = 15.39$, $ss(\text{diameter I}) = 41.14$, $ss(\text{diameter II}) = 16.0$, $ss(\text{diameter III}) = 15.12$, $ss(\text{diameter IV}) = 27.57$. The sum of square of usual random error ε is usually obtained by

subtracting all sums of squares of factorial effects having replication effect from total sum of square. Thus mean ss(error)=20.67

Summary

The construction, estimates and analysis of 3^n circular factorial designs are explained in detail. The design contains multidirectional data in different directional diameters and controls the multidirectional heterogeneity. Its estimates are more robust in comparison to those of other factorial designs of same size.

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