Abstract. This paper is dedicated to a prevailing during the last fifteen years approach to noncommutative algebraic geometry in which ‘spaces’ are identified with categories thought as categories of quasi-coherent or coherent sheaves. We introduce a universal for this viewpoint category of ‘spaces’ and develop basics of algebraic geometry inside of this category; i.e. we define affine objects, classes of morphisms which have geometric meaning, and glue locally affine spaces and schemes. One of the main tools for studying and describing thus defined locally affine spaces and morphisms between them is provided by a noncommutative version of flat descent.

Introduction

Spaces considered in commutative geometry, are either locally ringed topological spaces (otherwise called geometric spaces), or sheaves of sets on the site CAff of commutative affine schemes. Although these two categories of spaces have little in common, the categories of schemes obtained by glueing affine objects are naturally equivalent.

Noncommutative algebraic geometry is developing under influence of examples of interest which are felt to be of geometric nature, but appear in different surroundings. These examples suggest the choices of the category of ‘spaces’. Thus, noncommutative analogue of Proj is defined as a category [M1], [V1], or a category with a fixed object $\mathcal{O}$ (= a structure sheaf) [AZ]. Projective spaces introduced in [KR1] appear as functors from the category $\text{Alg}_k$ of associative unital $k$-algebras to $\text{Sets}$.

We studied ‘spaces’ represented by sheaves of sets in [KR2] and noncommutative stacks in [KR3]. Here we undertake an elementary study of ‘spaces’ represented by categories regarded as categories of quasi-coherent or coherent sheaves on (maybe ‘virtual’) spaces. Similarly to [KR2], one of the purposes of this work is to provide a detailed background for the construction of non-affine spaces sketched in [KR1]. The paper is turning around several examples and establishes (a part of) a dictionary which extends geometric notions to ‘abstract nonsense’ environment. This explains the length of the manuscript.

It is important to realize that the setting considered in this work is one of the faces of noncommutative geometry. There are several others. Among them, ‘spaces’ represented by triangulated categories, or DG-, or $A_{\infty}$-categories are of increasing importance and require a thorough investigation. We do not discuss them in this paper.

The category of ‘spaces’. The minimal starting data for a geometric theory consists of: a category, $\mathcal{B}$, of ‘spaces’, a category $\mathcal{A}$ of ‘local’ (or ‘affine’) objects, a functor from the category $\mathcal{A}$ to the category $\mathcal{B}$ which assigns to each affine object a ‘space’. Given a set of covers for all objects of $\mathcal{B}$, we define the notion of a locally affine ‘space’ (details of this formalism can be found in sections 9.1 – 9.5).

We add to this minimal data one more ingredient: a map which assigns to each ‘space’ $X$ a category, $C_X$, regarded as the category of quasi-coherent modules on $X$. New Prairie Press, 2014
This assignment is supposed to be a contravariant pseudo-functor from the category $\mathfrak{B}$ of 'spaces' to the category $\mathcal{C}at$ of categories (which belong to a given universum). In other words, we have a fibered category over the category $\mathfrak{B}$. There is a universal (in a sense which is made precise in the text) for this setting category of 'spaces', $|\mathcal{C}at|^o$. It has same objects as the category $\mathcal{C}at^{op}$ opposite to $\mathcal{C}at$, morphisms are isomorphism classes of 1-morphisms of $\mathcal{C}at^{op}$. The universal fibered category of 'spaces' is given by the functor from $\mathcal{C}at^{op}$ to $|\mathcal{C}at|^o$ which is identical on objects and assigns to each 1-morphism of $\mathcal{C}at^{op}$ its isomorphism class. We interpret 1-morphisms of $\mathcal{C}at^{op}$ as inverse image functors of the corresponding morphisms of $|\mathcal{C}at|^o$ (i.e morphisms of 'spaces').

This is the starting point of the work. The immediate purpose is to define and study geometry (or geometries) inside of the category $|\mathcal{C}at|^o$, like commutative scheme theory is defined and studied inside of the category of locally ringed spaces.

What we really study and use is not so much 'spaces', but certain classes of morphisms between 'spaces'. The most important among them are continuous', flat, and affine morphisms introduced in [R]. A morphism is continuous if its inverse image functor has a right adjoint (called a direct image functor), and flat if, in addition, the inverse image functor is left exact (i.e. preserves finite limits). A continuous morphism is called affine if its direct image functor is conservative (i.e. reflects isomorphisms) and has a right adjoint.

Affine and locally affine 'spaces'. Given an object $S$ of the category $|\mathcal{C}at|^o$, we define the category $\text{Aff}_S$ of affine $S$-spaces as the full subcategory of $|\mathcal{C}at|^o/S$ whose objects are affine morphisms to $S$. The functor from $\text{Aff}_S$ to $|\mathcal{C}at|^o$ is the composition of the inclusion functor $\text{Aff}_S \hookrightarrow |\mathcal{C}at|^o/S$ and the canonical functor $|\mathcal{C}at|^o/S \rightarrow |\mathcal{C}at|^o$. The choice of the object $S$ influences drastically the rest of the story. Thus, if $S = \text{Sp}\mathbb{Z}$ (i.e. $C_S$ is the category of abelian groups), then $\text{Aff}_S$ is equivalent to the category opposite to the category $\mathfrak{A}ss$ which is defined as follows: objects of $\mathfrak{A}ss$ are associative unital rings and morphisms are conjugation classes of unital ring morphisms.

If $C_S$ is the category $\text{Sets}$, then the category $\text{Aff}_S$ is equivalent to the category $\widetilde{\mathfrak{M}on}^{op}$, where objects of $\widetilde{\mathfrak{M}on}$ are monoids and morphisms are conjugation classes of monoid homomorphisms.

Locally affine objects are defined in a straightforward way, once a notion of a cover (a quasi-pretopology) is fixed. We introduce several canonical quasi-pretopologies on the category $|\mathcal{C}at|^o$. Their common feature is the following: if a set of morphisms to $X$ is a cover, then the set of their inverse image functors is conservative and all inverse image functors are exact in a certain mild way. If, in addition, morphisms of covers are continuous, $X$ has a finite affine cover, and the category $C_S$ has finite limits, this requirement suffices to recover the object $X$ from the covering data uniquely up to isomorphism (i.e. the category $C_X$ is recovered uniquely up to equivalence) via 'flat descent'.

Semiseparated covers and 'spaces'. We call a cover 'semiseparated' if it is finite and consists of affine morphisms. Semiseparated covers form a quasi-pretopology on $|\mathcal{C}at|^o$. We call a 'space' $X$ 'semiseparated' if it has a semiseparated cover by 'affine spaces' (i.e. images of objects of $\text{Aff}_S$). If the base 'space' $S$ is $\text{Sp}(R)$ for some associative unital ring $R$ (that is $C_S$ coincides with the category $R - mod$ of left $R$-modules), then semiseparated 'spaces' is the answer to the question 'how much of geometry linear algebra can incorporate?'. Explicitly, if $X$ is semiseparated over $\text{Sp}(R)$, then there exists a ring $T$ and...
a coalgebra $H$ in the category of $R$-bimodules such that the category $C_X$ is equivalent to the category of $H$-comodules, i.e. left $R$-modules with a coaction of $H$. This fact is a consequence of flat descent which is one of the topics and tools of this work.

Recall that a scheme $X$ is called semiseparated if it has an affine cover $\{U_i \hookrightarrow X \mid i \in I\}$ such that each morphism $U_i \hookrightarrow X$ is affine. Any separated scheme is semiseparated; in particular, all classical varieties are semiseparated schemes. One can show that a space $X$ (geometric or functorial) is a semiseparated scheme iff its image in $|\text{Cat}|^o$ (determined by pseudo-functor which assigns to a space $X$ the category $\text{Qcoh}_X$ of quasi-coherent sheaves on $X$) is a semiseparated scheme over $\text{Sp}\overline{Z}$ in the meaning above.

**Locally affine morphisms.** Even if we consider only semiseparated spaces and schemes over $\overline{Z}$ (i.e. $S = \text{Sp}\overline{Z}$), the geometry we obtain via the functor $\text{Aff}_S \longrightarrow |\text{Cat}|^o$ is unusual: our locally affine spaces do not have, in general, a global sections functor. Global sections functor appears if instead of $|\text{Cat}|^o$, we take the full subcategory $|\text{Cat}|^o_S$ of the category $|\text{Cat}|^o/S$ of $S$-‘spaces’ whose objects are pairs $(X, f)$, where $f$ is a continuous morphism $X \longrightarrow S$. The choice of the category of local objects is the same as above, i.e. $\text{Aff}_S$, and the functor from $\text{Aff}_S$ to $|\text{Cat}|^o_S$ is the inclusion. It follows from our results that this inclusion is a full (hence a fully faithful) functor. Quasi-pretopologies we introduced on $|\text{Cat}|^o$ induce quasi-pretopologies on $|\text{Cat}|^o_S$. Applying general formalism, we define locally affine ‘spaces’; using flat descent (in the case of covers formed by continuous morphisms), we obtain their description via local data.

We apply flat descent consideration to describing morphisms between locally affine ‘spaces’ in terms of given affine covers of these ‘spaces’. A demand to obtain such description immerged in our work [KR1]. Notice that due to the fact that flat covers do not form a pretopology (the base change invariance fails), the standard way to describe morphisms via covers is not available.

Besides of these conceptual facts, the work contains several other things. For instance, we give a useful description of continuous morphisms to the categoric spectrum of a ring. We introduce and study (in ‘Complementary facts’) noncommutative quasi-affine ‘spaces’ (spectra of non-unital associative rings) illustrated by some examples coming from representation theory.

The paper is organized as follows.

In Section 1, we introduce first notions of categoric geometry (‘spaces’ represented by categories, morphisms represented by their inverse image functors, continuous, flat and affine morphisms) and sketch several examples of noncommutative spaces which are among illustrations and motivations of constructions of this work.

In Section 2, we start to view the category $\text{Cat}^{\text{op}}$ opposite to the category of categories (which belong to a universum $\mathfrak{U}$) as the bicategory of ‘spaces’: objects are regarded as ‘spaces’, the corresponding categories as categories of quasi-coherent sheaves, and functors as inverse image functors of morphisms they define. We introduce the category of spaces, $|\text{Cat}|^o$, by identifying isomorphic inverse image functors. We observe that the natural functor $\text{Cat}^{\text{op}} \longrightarrow |\text{Cat}|^o$ (identical on objects) is a universal object in the category of fibered categories of certain type.

In Section 3, we introduce (or ruther recall the definition of) different classes of morphisms, starting with continuous, flat, and affine morphisms. Their meaning is illustrated
by examples. In particular, we consider a functor from the category $\mathbf{Aff}$ of affine (noncommutative) schemes defined as the category opposite to the category $\mathbf{Rings}$ of associative rings to the category $|\mathbf{Cat}|^o$ of ’spaces’ which assigns to each ring $R$ its categoric spectrum, $\text{Sp}(R)$ (corresponding to the category of left $R$-modules) and to each ring morphism an affine morphism between the respective categoric spectra.

In Section 4, we describe continuous morphisms from an arbitrary ”space” $X$ to the categoric spectrum, $\text{Sp}(R)$, of a ring $R$. We argue that continuous morphisms $X \to \text{Sp}(R)$ are in one-to-one correspondence with isomorphism classes of right $R$-modules $O$ in the category $C_X$ of quasi-coherent modules on $X$ (i.e. $R$-modules in the opposite category $C_X^{op}$). In the case $X = \text{Sp}(T)$ for some ring $T$, this correspondence expresses the classical fact (a theorem by Eilenberg and Moore) that inverse image functors of continuous morphisms $\text{Sp}(T) \to \text{Sp}(R)$ are given by $(T, R)$-bimodules.

In Section 5, we start to study continuous morphisms via monads and comonads associated with them, using as a main tool the Beck’s theorem characterizing the so called monadic and comonadic morphisms. For a monad $F$ on a ”space” $X$ (i.e. on the category $C_X$), we define the spectrum of $F$ as the ”space” $\text{Sp}(F/X)$ corresponding to the category of $F$-modules. The spectrum of a monad is a natural generalization of the categoric spectrum of a ring. Dually, for any comonad $G$ on $X$, we define its cospectrum, $\text{Sp}^o(X \setminus G)$ as the space corresponding to the category $G - \text{Comod}$ of $G$-comodules.

In Section 6, we exploit the fact that an affine morphism to $X$ is isomorphic to the canonical morphism $\text{Sp}(F/X) \to X$ for a continuous monad $F = (F, \mu)$. Here ’continuous’ means that the functor $F$ has a right adjoint. A consequence of this is that any affine morphism $Y \to \text{Sp}(R)$ is equivalent to the morphism $\text{Sp}(T) \to \text{Sp}(R)$ corresponding to a ring morphism $R \to T$. In particular, a direct image functor of any affine morphism $\text{Sp}(S) \to \text{Sp}(R)$ is a composition of a Morita equivalence and the ”restriction of scalars” (pull-back) functor corresponding to a ring morphism.

In Section 7, we study affine flat descent. If $U \xrightarrow{f} X$ is a flat conservative affine morphism (’conservative’ means that $f^*$ reflects isomorphisms), then it follows from Beck’s theorem that $X$ is isomorphic to $\text{Sp}^o(U \setminus G_f)$, where $G_f = (G_f, \delta_f)$ is a continuous comonad. ’Continuous’ means that the functor $G_f$ has a right adjoint. In the case $U = \text{Sp}(R)$ for some ring $R$, continuous comonads are given by coalgebras in the category of $R$-bimodules. The main commutative example is an arbitrary semiseparated quasi-compact scheme. Recall that a scheme $X$ is semiseparated iff it has an affine cover $\{U_i \hookrightarrow X \mid i \in J\}$ such that all embeddings $U_i \hookrightarrow X$ are affine morphisms.

Section 8 is dedicated to finiteness conditions (locally finitely presentable morphisms and objects) and smooth and étale morphisms. We start with a simple general formalism and then specialize it in the case of ’spaces’.

In Section 9, we introduce a general formalism of glueing locally affine objects and apply it to define locally affine spaces and schemes in the category $|\mathbf{Cat}|^o$ and in the categories $|\mathbf{Cat}|^o_S$ of $S$-’spaces’. Here, we define locally affine spaces and schemes in full generality. Note that noncommutative locally affine spaces and schemes defined in earlier works on the subject [R1], [KR] belong to the class of semiseparated quasi-compact commutative schemes.

Section 10 is dedicated to the flat descent of morphisms.
The content of the second part of the paper, Complementary facts, reflects its title. In Section C1, we give a description of continuous morphisms to the categoric spectrum of a monoid which is analogous to the description of continuous morphisms to the categoric spectrum of a ring obtained in Section 4.

In Section C2, we discuss compatibility of monads and continuous morphisms with localizations. The facts obtained here are used in the next 3 sections.

Section C3 complements and clarifies Example 1.6. Here we introduce and study the cone of a non-unital monad and, as a special case, the cone of a non-unital associative ring; the latter is regarded as a noncommutative quasi-affine ‘space’. Motivated by important constructions of representation theory of classical and quantum groups and enveloping algebras, we consider Hopf actions on non-unital rings and induced actions on the corresponding quasi-affine ‘spaces’.

In Section C4, we construct the Proj of graded non-unital monad. As an application, we recover the Proj of a graded associative ring introduced in Example 1.7.

In Section C5, we resume the study of Hopf actions started in C3. Applying general facts to the action of enveloping algebra of a semisimple (or reductive) Lie algebra we realize the category of D-modules on the base affine space and the flag variety as categories of quasi-coherent sheaves on resp. the cone and the Proj of a graded ring naturally associated with the Lie algebra.

This setting is extended to actions of the quantized enveloping algebra of a semisimple Lie algebra on the quantum base affine ‘space’ and the quantum flag variety. Altogether is a complement to 1.10.

In Section C6 (which is a complement ot Sections 7 and 9), we study affine flat covers.

In Section C7 (complementing Section 10), we study descent of flat morphisms.

In Sections C8 and C9, we study connection between flat descent and the descent procedure using relations.

Section C10 gathers miscellaneous facts, some of them suggestive.

We introduce connections and interpret comonads as integrable connections.

We introduce weakly quasi-affine morphisms and relatively ample morphisms.

We discuss continuous, flat, and affine morphisms in the fibered category of modules on affine (noncommutative) schemes.

Finally, we complement duality of Section 6 (based on the connection between continuous monads and affine morphisms) by producing conditions which garantee the existence of a canonical functor which assigns to each (additive, or arbitrary) monad a continuous monad.

A part of this paper was written while the second author was visiting Max-Planck Institute für Mathematik in Bonn during the summer of 2001 and provided topics for several talks given at the seminar on noncommutative geometry at MPI. Special thanks to Yu. I. Manin for useful conversations, for proposing these lectures, and for being a stimulating listener. It is a pleasure to acknowledge excellent working conditions provided by the Institute.

The work of the second author on this paper was partially supported by the NSF grant DMS-0070921.
1. 'Spaces' and categories. Examples.

1.1. Categories and 'spaces'. As usual, \( \text{Cat} \), or \( \text{Cat}_\mathcal{U} \), denotes the bicategory of categories which belong to a fixed universum \( \mathcal{U} \). We call objects of \( \text{Cat}^{\text{op}} \) 'spaces'. For any 'space' \( X \), the corresponding category \( \mathcal{C} \) is regarded as the category of quasi-coherent sheaves on \( X \). We denote by \( f^* \) the functor \( \mathcal{C}_Y \to \mathcal{C}_X \) corresponding to a 1-morphism \( X \to Y \) and call it the inverse image functor of \( f \). For any \( \mathcal{U} \)-category \( \mathcal{A} \), we denote by \( \mathcal{A} \) the corresponding object of \( \text{Cat}^{\text{op}} \) (the underlying 'space') defined by \( \mathcal{C}_{\mathcal{A}} = \mathcal{A} \).

We denote by \( \mathcal{C} \) the category having same objects as \( \text{Cat}^{\text{op}} \). Morphisms from \( X \) to \( Y \) are isomorphism classes of functors \( \mathcal{C}_Y \to \mathcal{C}_X \). For a morphism \( X \to Y \), we denote by \( f^* \) any functor \( \mathcal{C}_Y \to \mathcal{C}_X \) representing \( f \) and call it an inverse image functor of the morphism \( f \). We shall write \( f = [F] \) to indicate that \( f \) is a morphism having an inverse image functor \( F \). The composition of morphisms \( X \to Y \) and \( Y \to Z \) is defined by \( g \circ f = [f^* \circ g^*] \). Thus, the map which assigns to each functor \( \mathcal{C}_Y \to \mathcal{C}_X \) the morphism \( X \to Y \) is a functor \( \text{Cat}^{\text{op}} \to \mathcal{C} \) which turns \( \text{Cat}^{\text{op}} \) into a fibred category over \( \mathcal{C} \).

1.2. Localizations and conservative morphisms. Let \( Y \) be an object of \( \mathcal{C} \) and \( \Sigma \) a class of arrows of the category \( \mathcal{C}_Y \). We denote by \( \Sigma^{-1}Y \) the object of \( \mathcal{C} \) such that the corresponding category coincides with (the standard realization of) the quotient of the category \( \mathcal{C}_Y \) by \( \Sigma \) (cf. [GZ, 1.1]): \( \mathcal{C}_{\Sigma^{-1}Y} = \Sigma^{-1}\mathcal{C}_Y \). The canonical localization functor \( \mathcal{C}_Y \to \Sigma^{-1}\mathcal{C}_Y \) is regarded as an inverse image functor of a morphism, \( \Sigma^{-1}Y \to Y \).

For any morphism \( X \to Y \) in \( \mathcal{C} \), we denote by \( \Sigma_f \) the family of all arrows \( s \) of the category \( \mathcal{C}_Y \) such that \( f^*(s) \) is invertible (notice that \( \Sigma_f \) does not depend on the choice of an inverse image functor \( f^* \)). Thanks to the universal property of localizations, \( f^* \) is represented as the composition of the localization functor \( p_f^* = p_{\Sigma_f}^*: \mathcal{C}_Y \to \Sigma_f^{-1}\mathcal{C}_Y \)

and a uniquely determined functor \( \Sigma_f^{-1}\mathcal{C}_Y \to \mathcal{C}_X \). In other words, \( f = p_f \circ f_c \) for a uniquely determined morphism \( X \to \Sigma_f^{-1}Y \).

A morphism \( X \to Y \) is called conservative if \( \Sigma_f \) consists of isomorphisms only, or, equivalently, \( p_f \) is an isomorphism.

A morphism \( X \to Y \) is called a localization if \( f_c \) is an isomorphism, i.e. the functor \( f_c^* \) is an equivalence of categories.

Thus, \( f = p_f \circ f_c \) is a unique decomposition of a morphism \( f \) into a localization and a conservative morphism.

1.3. Continuous, flat, and affine morphisms. A morphism is called continuous if its inverse image functor has a right adjoint (called a direct image functor), and flat if, in addition, the inverse image functor is left exact (i.e. preserves finite limits). A continuous morphism is called 'affine' if its direct image functor is conservative (i.e. reflects isomorphisms) and has a right adjoint.

1.4. Categoric spectrum of a unital ring. For an associative unital ring \( R \), we define the categoric spectrum of \( R \) as the object \( \text{Sp}(R) \) of \( \mathcal{C} \) such that \( \mathcal{C}_{\text{Sp}(R)} = \)
Let \( R \xrightarrow{\phi} S \) be a unital ring morphism and \( R \xrightarrow{-mod} S \xrightarrow{\phi^*} -mod \) the functor \( S \otimes_R - \). The canonical right adjoint to \( \phi^* \) is the pull-back functor by the ring morphism \( \phi \). A right adjoint to \( \phi_* \) is given by

\[
\phi^!: S -mod \longrightarrow R -mod, \quad L \longmapsto Hom_R(\phi_*(S), L).
\]

The map

\[
\left( R \xrightarrow{\phi} S \right) \longmapsto \left( \text{Sp}(S) \xrightarrow{\phi} \text{Sp}(R) \right)
\]

is a functor

\[
\text{Sp}: \text{Rings}^{op} \longrightarrow |\text{Cat}|^{o}
\]

which takes values in the subcategory formed by affine morphisms.

The image \( \text{Sp}(R) \xrightarrow{\phi} \text{Sp}(T) \) of a ring morphism \( T \xrightarrow{\phi} R \) is flat (resp. faithful) iff \( \phi \) turns \( R \) into a flat (resp. faithful) right \( T \)-module.

1.4.1. Continuous, flat, and affine morphisms from \( \text{Sp}(S) \) to \( \text{Sp}(R) \). Let \( R \) and \( S \) be associative unital rings. A morphism \( f: \text{Sp}(S) \longrightarrow \text{Sp}(R) \) with an inverse image functor \( f^* \) is continuous iff

\[
f^* \simeq M \otimes_R : L \longmapsto M \otimes_R L \quad (1)
\]

for an \((S, R)\)-bimodule \( M \) defined uniquely up to isomorphism. The functor

\[
f_* = Hom_S(M, -): N \longmapsto Hom_S(M, N) \quad (2)
\]

is a direct image of \( f \).

The morphism \( f \) with an inverse image functor (1) is conservative iff \( M \) is faithful as a right \( R \)-module, i.e. the functor \( M \otimes_R - \) is faithful.

The direct image functor (2) is conservative iff \( M \) is a cogenerator in the category of left \( S \)-modules, i.e. for any nonzero \( S \)-module \( N \), there exists a nonzero \( S \)-module morphism \( M \longrightarrow N \).

The morphism \( f \) is flat iff \( M \) is flat as a right \( R \)-module.

The functor (2) has a right adjoint, \( f^! \), iff \( f_* \) is isomorphic to the tensoring (over \( S \)) by a bimodule. This happens iff \( M \) is a projective \( S \)-module of finite type. The latter is equivalent to the condition: the natural functor morphism \( M_\ast \otimes_S - \longrightarrow Hom_S(M, -) \) is an isomorphism. Here \( M_\ast = Hom_S(M, S) \). In this case, \( f^! \simeq Hom_R(M_\ast, -) \).

1.5. The graded version. Let \( G \) be a monoid and \( R \) a \( G \)-graded unital ring. We define the ‘space’ \( \text{Sp}_G(R) \) by taking as \( C_{\text{Sp}_G(R)} \) the category \( \text{gr}_G R \xrightarrow{-mod} \) of left \( G \)-graded \( R \)-modules. There is a natural functor \( \text{gr}_G R \xrightarrow{-mod} R_0 \xrightarrow{-mod} \) which assigns to each graded \( R \)-module its zero component (‘zero’ is the unit element of the monoid \( G \)). The functor \( \phi_* \) has a left adjoint, \( \phi^* \), which maps every \( R_0 \)-module \( M \) to the graded \( R \)-module \( R \otimes_{R_0} M \). The adjunction arrow \( Id_{R_0 \xrightarrow{-mod} R_0} \longrightarrow \phi_* \phi^* \) is an isomorphism. This means that the functor \( \phi^* \) is fully faithful, or, equivalently, the functor \( \phi_* \) is a localization.
The functors $\phi_*$ and $\phi^*$ are regarded as respectively a direct and an inverse image functor of a morphism $\mathbf{Sp}_G(R) \xrightarrow{\phi} \mathbf{Sp}(R_0)$. It follows from the above that the morphism $\phi$ is affine iff $\phi$ is an isomorphism (i.e. $\phi^*$ is an equivalence of categories).

In fact, if $\phi$ is affine, the functor $\phi_*$ should be conservative. Since $\phi_*$ is a localization, this means, precisely, that $\phi_*$ is an equivalence of categories.

1.6. The cone of a non-unital ring. Let $R_0$ be a unital associative ring, and let $R_+$ be an associative ring, non-unital in general, in the category of $R_0$-bimodules; i.e. $R_+$ is endowed with an $R_0$-bimodule morphism $R_+ \otimes R_0 R_+ \xrightarrow{m} R_+$ satisfying the associativity condition. Let $R = R_0 \oplus R_+$ denote the augmented ring described by this data. Let $T_{R_+}$ denote the full subcategory of the category $R - \text{mod}$ whose objects are all $R$-modules annihilated by $R_+$. Let $T_{R_+}^-$ be the Serre subcategory (that is a full subcategory closed by taking subquotients, extensions, and arbitrary direct sums) of the category $R - \text{mod}$ spanned by $T_{R_+}$.

We define the 'space' cone of $R_+$ by taking as $C_{\text{Cone}(R_+)}$ the quotient category $R - \text{mod}/T_{R_+}^-$. The localization functor $R - \text{mod} \xrightarrow{u^*} R - \text{mod}/T_{R_+}^-$ is an inverse image functor of a morphism of 'spaces' $\text{Cone}(R_+) \xrightarrow{u} \mathbf{Sp}(R)$. The functor $u^*$ has a (necessarily fully faithful) right adjoint, i.e. the morphism $u$ is continuous. If $R_+$ is a unital ring, then $u$ is an isomorphism (see C3.2.1). The composition of the morphism $u$ with the canonical affine morphism $\mathbf{Sp}(R) \xrightarrow{\phi} \mathbf{Sp}(R_0)$ is a continuous morphism $\text{Cone}(R_+) \xrightarrow{\phi} \mathbf{Sp}(R_0)$. Its direct image functor is (regarded as) the global sections functor.

1.7. The graded version: Proj$_G$. Let $G$ be a monoid and $R = R_0 \oplus R_+$ a $G$-graded ring with zero component $R_0$. Then we have the category $gr_G R - \text{mod}$ of $G$-graded $R$-modules and its full subcategory $gr_G T_{R_+} = T_{R_+} \cap gr_G R - \text{mod}$ whose objects are graded modules annihilated by the ideal $R_+$. We define the 'space' Proj$_G(R)$ by setting

$$C_{\text{Proj}_G(R)} = gr_G R - \text{mod}/gr_G T_{R_+}^-.$$ 

Here $gr_G T_{R_+}^-$ is the Serre subcategory of the category $gr_G R - \text{mod}$ spanned by $gr_G T_{R_+}^-$. One can show that $gr_G T_{R_+}^- = gr_G R - \text{mod} \cap T_{R_+}^-$. Therefore, we have a canonical projection

$$\text{Cone}(R_+) \xrightarrow{\phi} \text{Proj}_G(R).$$

The localization functor $gr_G R - \text{mod} \xrightarrow{\phi} C_{\text{Proj}_G(R)}$ is an inverse image functor of a continuous morphism $\text{Proj}_G(R) \xrightarrow{\phi} \mathbf{Sp}_G(R)$. The composition $\text{Proj}_G(R) \xrightarrow{\phi} \mathbf{Sp}(R_0)$ of the morphism $\phi$ with the canonical morphism $\mathbf{Sp}_G(R) \xrightarrow{\phi} \mathbf{Sp}(R_0)$ defines Proj$_G(R)$ as a 'space' over $\mathbf{Sp}(R_0)$. Its direct image functor is called the global sections functor.

1.7.1. Example: cone and Proj of a $\mathbb{Z}_+$-graded ring. Let $R = \bigoplus_{n \geq 0} R_n$ be a $\mathbb{Z}_+$-graded ring, $R_+ = \bigoplus_{n \geq 1} R_n$ its 'irrelevant' ideal. Thus, we have the cone of $R_+$, Cone$(R_+)$, and Proj$(R) = \text{Proj}_G(R)$, and a canonical morphism Cone$(R_+) \xrightarrow{\phi} \text{Proj}(R)$.

1.8. Example: skew cones and skew projective 'spaces'. Let $A$ be an arbitrary associative $k$-algebra. And let $q$ denote a matrix $[q_{ij}]_{i,j \in J}$ with entrees in $k$ such that
\(q_{ij}q_{ji} = 1\) for all \(i, j \in J\) and \(q_{ii} = 1\) for all \(i \in J\). Let \(R = A_q[x]\) denote a skew polynomial algebra corresponding to this data. Here \(x = (x_i \mid i \in J)\) is a set of indeterminates satisfying the relations

\[
\begin{align*}
x_i x_j &= q_{ij} x_j x_i & \text{for all } i, j \in J, \\
x_i r &= rx_i & \text{for all } i \in J \text{ and } r \in R
\end{align*}
\]

For any \(i \in J\), set \(S_i = \{x_i^n \mid n \geq 1\}\). Each of \(S_i\) is a left and right Ore set, and the Serre subcategory \(T_{R_i}^-\) is generated by \(R\)-modules whose elements are annihilated by some elements of \(\bigcup_{i \in J} S_i\). This implies that the localizations \(R - \text{mod} \to S_i^{-1}R - \text{mod}\) factor through the localization \(R - \text{mod} \to \text{Cone}(R_+)\), and the induced localizations \(\text{Cone}(R_+) \xrightarrow{u_i} S_i^{-1}R - \text{mod}\) form a conservative family. The conservative family \(\{\text{Sp}(S_i^{-1}R) \to \text{Cone}(R_+) \mid i \in J\}\) is regarded as an affine cover of the cone \(\text{Cone}(R_+)\). It follows that the algebra \(S_i^{-1}R\) is isomorphic to \(A_q[x, x_i^{-1}]\).

Let \(G = \mathbb{Z}^J\); and let \(\gamma_i, i \in J\), denote the canonical generators of the group \(G\). Assigning to each \(x_i\) the parity \(\gamma_i\), we turn the skew polynomial algebra \(R = A_q[x]\) into a \(G\)-graded algebra with \(R_0 = A\). The localizations \(R - \text{mod} \to S_i^{-1}R - \text{mod}\) induce localizations \(gr_G R - \text{mod} \to gr_G S_i^{-1}R - \text{mod}\) which factor through the localization

\[
C_{\text{Proj}(R)} \xrightarrow{v_i} gr_G S_i^{-1}R - \text{mod} = gr_G A_q[x, x_i^{-1}] - \text{mod}. \tag{3}
\]

Let \(G_i\) denote the quotient group \(G/\mathbb{Z}\gamma_i\). The category \(gr_G A_q[x, x_i^{-1}] - \text{mod}\) in (3) is naturally equivalent to the category \(gr_G A_q[x/x_i] - \text{mod}\) of left \(G_i\)-graded modules over the skew polynomial algebra \(A_q[x/x_i]\). Here \(x/x_i\) denotes \(\{x_j/x_i \mid j \in J, j \neq i\}\), and \(q_i\) denotes the matrix \([q_{nij}q_{nmi}^{-1}]_{n, m \in J \setminus \{i\}}\) (cf. [R, I.7.2.2.4]). Note that \(A_q[x/x_i]\) is the \(G_i\)-component of the algebra \(A_q[x, x_i^{-1}]\) of the 'functions' on \(\text{Cone}(R)/|S_i|\).

Let 'spaces' \(U_i\) are defined by \(C_{U_i} = gr_G A_q[x/x_i] - \text{mod}\). Note that if the cardinality of \(J\) is greater than one, then the natural morphisms \(U_i \xrightarrow{u_i} \text{Proj}_G(R)\) do not form an affine cover of \(\text{Proj}_G(R)\) over \(\text{Sp}(A)\), because the composition of \(u_i\) with the direct image of the projection \(\text{Proj}_G(R) \xrightarrow{\pi} \text{Sp}(A)\) is isomorphic to the functor \(gr_G A_q[x/x_i] - \text{mod} \to A - \text{mod}\) which assigns to each \(G_i\)-graded module (resp. \(G_i\)-graded module morphism) its zero component. If the group \(G_i\) is non-trivial, this functor is not faithful, hence the morphism \(\pi \circ u_i\) is not affine.

1.8.1. The projective q-space. Let again \(R = A_q[x]\), \(x = (x_0, x_1, \ldots, x_r)\). But, we take \(G = \mathbb{Z}\) with the natural order; and set the parity of each \(x_i\) equal to 1. One can repeat with \(\text{Cone}_G(R)\) and \(P^r_G = \text{Proj}_G(R)\) the same pattern as with \(\text{Cone}(R)\) and \(P^r = \text{Proj}(R)\). Only this time the quotient groups \(G_i\) will be trivial, and we obtain a picture very similar to the classical one: \(P^r_q\) is a \(\mathbb{Z}\)-scheme covered by \(r + 1\) affine spaces \(A_q[x/x_i] - \text{mod}\), \(i = 0, 1, \ldots, r\).

1.9. The base affine space and the flag variety of a reductive Lie algebra. Let \(g\) be a reductive Lie algebra over \(\mathbb{C}\) and \(U(g)\) the enveloping algebra of \(g\). Let \(G\) be the group of integral weights of \(g\) and \(G_+\) the semigroup of nonnegative integral weights. Let \(R = \oplus_{\lambda \in G_+} R_\lambda\), where \(R_\lambda\) is the vector space of the (canonical) irreducible finite
dimensional representation with the highest weight $\lambda$. The module $R$ is a $G$-graded algebra with the multiplication determined by the projections $R_{\lambda} \otimes R_{\nu} \rightarrow R_{\lambda+\nu}$, for all $\lambda, \nu \in G_+$. It is well known that the algebra $R$ is isomorphic to the algebra of regular functions on the base affine space of $g$. Recall that $G/U$, where $G$ is a connected simply connected algebraic group with the Lie algebra $g$, and $U$ is its maximal unipotent subgroup.

The category $C_{\text{Cone}(R)}$ is equivalent to the category of quasi-coherent sheaves on the base affine space $Y$ of the Lie algebra $g$. The category $\text{Proj}_G(R)$ is equivalent to the category of quasi-coherent sheaves on the flag variety of $g$.

### 1.10. The quantized base affine 'space' and quantized flag variety of a semisimple Lie algebra.

Let now $g$ be a semisimple Lie algebra over a field $k$ of zero characteristic, and let $U_q(g)$ be the quantized enveloping algebra of $g$. Define the $G$-graded algebra $R = \bigoplus_{\lambda \in G_+} R_{\lambda}$ the same way as above. This time, however, the algebra $R$ is not commutative. Following the classical example (and identifying spaces with categories of quasi-coherent sheaves on them), we call $\text{Cone}(R)$ the quantum base affine 'space' and $\text{Proj}_G(R)$ the quantum flag variety of $g$.

#### 1.10.1. Canonical affine covers of the base affine 'space' and the flag variety.

Let $W$ be the Weyl group of the Lie algebra $g$. Fix a $w \in W$. For any $\lambda \in G_+$, choose a nonzero $w$-extremal vector $e_{w\lambda}$ generating the one dimensional vector subspace of $R_{\lambda}$ formed by the vectors of the weight $w\lambda$. Set $S_w = \{k^s e_{w\lambda}| \lambda \in G_+\}$. It follows from the Weyl character formula that $e_{w\lambda} e_{w\mu} \in k^s e_{w(\lambda+\mu)}$. Hence $S_w$ is a multiplicative set. It was proved by Joseph [Jo] that $S_w$ is a left and right Ore subset in $R$. The Ore sets $\{S_w| w \in W\}$ determine a conservative family of affine localizations

$$\text{Sp}(S_w^{-1}R) \longrightarrow \text{Cone}(R), \quad w \in W,$$

of the quantum base affine 'space' and a conservative family of affine localizations

$$\text{Sp}_G(S_w^{-1}R) \longrightarrow \text{Proj}_G(R), \quad w \in W,$$

of the quantum flag variety. We claim that the category $gr G S_w^{-1}R - \text{mod}$ is naturally equivalent to $(S_w^{-1}R)_0 - \text{mod}$. By 1.5, it suffices to verify that the canonical functor $gr G S_w^{-1}R - \text{mod} \longrightarrow S_w^{-1}R_0 - \text{mod}$ which assigns to every graded $S_w^{-1}R$-module its zero component is faithful; i.e. the zero component of every nonzero $G$-graded $S_w^{-1}R$-module is nonzero. This is, really, the case, because if $z$ is a nonzero element of $\lambda$-component of a $G$-graded $S_w^{-1}R$-module, then $(e_{w\lambda}^\lambda)^{-1}z$ is a nonzero element of the zero component of this module.

#### 1.11. Noncommutative Grassmannians and projective spaces.

Fix an associative unital $k$-algebra $R$. Let $R \backslash \text{Alg}_k$ be the category of associative $k$-algebras over $R$ (i.e. pairs $(S, R \rightarrow S)$, where $S$ is a $k$-algebra and $R \rightarrow S$ a $k$-algebra morphism). We call them for convenience $R$-rings. We denote by $R^o$ the $k$-algebra $R \otimes_k R^o$. Here $R^o$ is the algebra opposite to $R$.

#### 1.11.1. The functor $Gr_{M,V}$.

Let $M$, $V$ be left $R$-modules. Consider the functor, $Gr_{M,V} : R \backslash \text{Alg}_k \longrightarrow \text{Sets}$, which assigns to any $R$-ring $(S, R \rightarrow S)$ the set of isomorphism
classes of epimorphisms $s^*(M) \rightarrow s^*(V)$ (here $s^*(M) = S \otimes_R M$) and to any $R$-ring morphism $(S, R \xrightarrow{ \phi } S) \rightarrow (T, R \xrightarrow{t} T)$ the map $Gr_{M,V}(S,s) \rightarrow Gr_{M,V}(T,t)$ induced by
the inverse image functor $S \rightarrow mod \xrightarrow{\phi^*} T \rightarrow mod$, $N \mapsto T \otimes_SN$.

1.11.2. The functor $G_{M,V}$. Denote by $G_{M,V}$ the functor $R \setminus \text{Alg}_k \rightarrow \text{Sets}$ which
assigns to any $R$-ring $(S, R \xrightarrow{s} S)$ the set of all 4-tuples $(u_1, v_1; u_2, v_2) \in G_{M,V} \times G_{M,V}$ such that the epimorphisms $u_1, u_2$ are equivalent. The latter means that there exists an isomorphism $s^*(V) \xrightarrow{\varphi} s^*(V)$ such that $u_2 = \varphi \circ u_1$, or, equivalently, $\varphi^{-1} \circ u_2 = u_1$. Since $u_i \circ v_i = id$, $i = 1, 2$, these equalities imply that $\varphi = u_2 \circ v_1$ and $\varphi^{-1} = u_1 \circ v_2$. Thus, $\mathcal{R}_{M,V}(S,s)$ is a subset of all $(u_1, v_1; u_2, v_2) \in G_{M,V}(S,s) \times G_{M,V}(S,s)$ satisfying the following relations:

$$u_2 = (u_2 \circ v_1) \circ u_1, \quad u_1 = (u_1 \circ v_2) \circ u_2$$

in addition to the relations describing $G_{M,V}(S,s) \times G_{M,V}(S,s)$:

$$u_1 \circ v_1 = id_{S \otimes_R V} = u_2 \circ v_2$$

Denote by $p_1, p_2$ the canonical projections $\mathcal{R}_{M,V} \xrightarrow{p_1} G_{M,V}$. It follows from the
surjectivity of $G_{M,V} \rightarrow Gr_{M,V}$ that the diagram

$$\mathcal{R}_{M,V} \xrightarrow{p_1} G_{M,V} \xrightarrow{\pi} Gr_{M,V}$$

is exact.

1.11.4. Proposition. If both $M$ and $V$ are projective modules of a finite type, then the functors $G_{M,V}$ and $\mathcal{R}_{M,V}$ are corepresentable.

Proof. See [KR2, 10.4.3]. □

1.11.5. Quasi-coherent presheaves on $Gr_{M,V}$. Suppose that $M$ and $V$ are projective modules of a finite type, hence the functors $G_{M,V}$ and $\mathcal{R}_{M,V}$ are corepresentable by $R$-rings resp. $R \rightarrow \mathfrak{S}_{M,V}$ and $(R \rightarrow \mathcal{R}_{M,V})$. Then the category $Qcoh(G_{M,V})$ (resp. $Qcoh(\mathcal{R}_{M,V})$) is equivalent to $\mathfrak{S}_{M,V} - mod$ (resp. $\mathcal{R}_{M,V} - mod$), and the category $Qcoh(Gr_{M,V})$ of quasi-coherent presheaves on $Gr_{M,V}$ is equivalent to the kernel of the diagram

$$Qcoh(G_{M,V}) \xrightarrow{p_1^*} Qcoh(\mathcal{R}_{M,V}) \xrightarrow{p_2^*} Qcoh(Gr_{M,V})$$
This means that, after identifying categories of quasi-coherent presheaves in (5) with corresponding categories of modules, quasi-coherent presheaves on $Gr_{M,V}$ can be realized as pairs $(L, \phi)$, where $L$ is a $\mathcal{G}_{M,V}$-module and $\phi$ is an isomorphism $p_{1}^{*}(L) \xrightarrow{\sim} p_{2}^{*}(L)$. Morphisms $(L, \phi) \to (N, \psi)$ are given by morphisms $L \xrightarrow{g} N$ such that the diagram

$$
p_{1}^{*}(L) \xrightarrow{p_{1}^{*}(g)} p_{1}^{*}(N) \quad \phi \downarrow \quad \downarrow \psi 
p_{2}^{*}(L) \xrightarrow{p_{2}^{*}(g)} p_{2}^{*}(N)$$

commutes. The functor

$$Qcoh(Gr_{M,V}) \xrightarrow{\pi^{*}} Qcoh(G_{M,V}), \quad (L, \phi) \mapsto L,$$

is an inverse image functor of the projection $G_{M,V} \xrightarrow{\pi} Gr_{M,V}$ (see 1.11.3(4))

1.11.6. Noncommutative Grassmannians. Let $\mathcal{S}$ be a quasi-topology on the category $(R \backslash Alg_{k})^{\text{op}}$ of affine $k$-schemes over $R$. Let $Gr_{M,V}^{\mathcal{S}}$ be the $\mathcal{S}$-Grassmannian corresponding to $Gr_{M,V}$, i.e. a sheaf of sets (a 'space') associated to the presheaf $Gr_{M,V}$. By [KR3, 2.9.1], if $\mathcal{S}$ is coarser than the quasi-topology of effective descent, then the category $Qcoh(Gr_{M,V}^{\mathcal{S}})$ of quasi-coherent modules on $Gr_{M,V}^{\mathcal{S}}$ is naturally equivalent to the category $Qcoh(Gr_{M,V})$ of quasi-coherent modules on the Grassmannian $Gr_{M,V}$.

1.11.7. Noncommutative projective space. Let $M$ be the free $R$-module of the rank $n+1$, $V$ the free $R$-module of the rank 1. In this case, we denote the functor $Gr_{M,V}$ by $P^{n}_{R}$. If a quasi-topology $\mathcal{S}$ on the category $(R \backslash Alg_{k})^{\text{op}}$ of affine $k$-schemes over $R$ is coarser than the quasi-topology of effective descent, than the category $Qcoh(P^{n}_{R})$ of quasi-coherent modules on $P^{n}_{R}$ is equivalent to the category of quasi-coherent modules on the (associated) projective space $\mathbb{P}^{n}_{R} = P^{n}_{R}^{\mathcal{S}}$.

2. Fibered categories and 'spaces'. Quasi-coherent modules.

2.1. Initial objects of $|\mathcal{C}at|^{o}$. The category $\bullet$ with one (identical) morphism (in particular with one object) is an initial object of $|\mathcal{C}at|^{o}$. A morphism $f : A \to B$ in $|\mathcal{C}at|^{o}$ with an inverse image functor $f^{*}$ is an isomorphism iff $f^{*}$ is a category equivalence. In particular, $X \in Ob|\mathcal{C}at|^{o}$ is an initial object of $|\mathcal{C}at|^{o}$ iff the category $C_{X}$ is a connected groupoid; i.e. $\text{Hom}C_{X}$ consists of isomorphisms and for any $x, y \in ObC_{X}$, the set $\text{Hom}_{C_{X}}(x, y)$ is non-empty.

Notice that for any object $X$ of $|\mathcal{C}at|^{o}$, the set $|\mathcal{C}at|^{o}(X, \bullet)$ of morphisms $X \to \bullet$ is isomorphic to the set $|X|$ of isomorphism classes of objects of the category $C_{X}$.

The category $|\mathcal{C}at|^{o}$ has no "real" final objects: the empty category is its unique final object.

2.2. Proposition. The category $|\mathcal{C}at|^{o}$ has small limits and colimits.
Proof. (a) Let \( \{X_i \mid i \in J\} \) be a set of objects of \(|\mathbf{Cat}|^\circ\). Then \( \prod_{i \in J} X_i \) and \( \prod_{i \in J} X_i \) are defined by
\[
\prod_{i \in J} X_i = \prod_{i \in J} C_{X_i} \quad \text{and} \quad \prod_{i \in J} X_i = \prod_{i \in J} C_{X_i}.
\]

(b) Every pair of arrows, \( X \xrightarrow{f} Y \), in \(|\mathbf{Cat}|^\circ\) has a cokernel.

Let \( f^*, g^* : C_Y \longrightarrow C_X \) be inverse image functors of resp. \( f \) and \( g \). Let \( C_Z \) denote the category whose objects are pairs \((x, \phi)\), where \( x \in \text{Ob} C_Y \) and \( \phi \) is an isomorphism \( f^*(x) \simeq g^*(x) \). A morphism from \((x, \phi)\) to \((y, \psi)\) is a morphism \( \xi : x \rightarrow y \) such that the diagram
\[
\begin{array}{ccc}
f^*(x) & \xrightarrow{f^*(\xi)} & f^*(y) \\
\phi & \downarrow & \downarrow \psi \\
f^*(x) & \xrightarrow{f^*(\xi)} & f^*(y)
\end{array}
\]
commutes. Denote by \( \mathfrak{h}^* \) the forgetful functor \( C_Z \longrightarrow C_Y \), \((x, \phi) \longmapsto x \). Let \( w : Y \longrightarrow W \) be a morphism in \(|\mathbf{Cat}|^\circ\) with an inverse image functor \( w^* \) such that \( w \circ f = w \circ g \). This means that there exists an isomorphism \( f^* \circ w^* \simeq g^* \circ w^* \). The pair \((w^*, \psi)\) defines a functor \( \gamma_{w^*, \psi} : C_W \longrightarrow C_Z \), \( b \longmapsto (w^*(b), \psi(b)) \). A different choice, \( w_1^* \), of the inverse image functor of \( w \) and an isomorphism \( \psi_1 : w_1^* \circ f^* \simeq g^* \circ w_1^* \) produces a functor \( \gamma_{w_1^*, \psi_1} \) isomorphic to \( \gamma_{w^*, \psi} \). This shows that the morphism \( Y \longrightarrow Z \) having the inverse image \( \mathfrak{h}^* \) is the cokernel of the pair \((f, g)\). The existence of cokernels and (small) coproducts is equivalent to the existence of arbitrary (small) colimits.

(c) Every pair of arrows, \( X \xrightarrow{f} Y \), in \(|\mathbf{Cat}|^\circ\) has a kernel.

Let \( f^*, g^* : C_Y \longrightarrow C_X \) be inverse image functors of resp. \( f \) and \( g \). Denote by \( \mathfrak{D}_{f^*, g^*} \) the diagram scheme defined as follows:
\[
\text{Ob} \mathfrak{D}_{f^*, g^*} = \text{Ob} C_Y \boxplus \text{Ob} C_X \quad \text{and} \quad \text{Hom} \mathfrak{D}_{f^*, g^*} = \text{Hom} C_X \boxplus \Sigma_{f^*, g^*},
\]
where
\[
\Sigma_{f^*, g^*} = \{ f^*(x) \xrightarrow{\alpha} x, \ x \xrightarrow{\beta} g^*(x) \mid x \in \text{Ob} C_Y \}.
\]
Consider the category \( \mathcal{P}a \mathfrak{D}_{f^*, g^*} \) of paths of the diagram \( \mathfrak{D}_{f^*, g^*} \) together with the natural embeddings \( \text{Hom} C_X \xrightarrow{\tau} \text{Hom} \mathcal{P}a \mathfrak{D}_{f^*, g^*} \xleftarrow{\Sigma_{f^*, g^*}} \), which define the corresponding diagrams. We denote by \( \mathcal{P} \mathfrak{D}_{f^*, g^*} \) the quotient of the category \( \mathcal{P}a \mathfrak{D}_{f^*, g^*} \) by the minimal equivalence relation such that
\[
\tau(\alpha \circ \beta) \sim \tau(\alpha) \circ \tau(\beta) \quad \text{and} \quad \tau(id_x) \sim id_{\tau(x)}
\]
for all composable arrows \( \alpha, \beta \) and for all \( x \in \text{Ob} C_X \).

Finally, we denote by \( C_W \) the quotient category \( \Sigma_{f^*, g^*}^{-1} \mathcal{P} \mathfrak{D}_{f^*, g^*} \). It follows from the construction that the object \( W \) of the category \(|\mathbf{Cat}|^\circ\) defined this way is the kernel of the pair \((f, g)\). Details are left to the reader.
2.3. Two fibered categories associated with Cat.

2.3.1. The fibered category \( \text{Cat}^o, \pi \) over \( \text{Cat}^{op} \). Consider the category \( \text{Cat}^o \) whose objects are pairs \((X, M)\), where \( X \in \text{ObCat}^{op} \) and \( M \in \text{Ob} \text{Cat}_X \). Morphisms from \((X, M)\) to \((Y, L)\) are pairs \((f, \xi)\), where \( f \) is a morphism \( X \rightarrow Y \) and \( \xi \) is a morphism \( f^*(L) \rightarrow M \). The pair \((\text{Cat}^o, \pi)\), where \( \pi: \text{Cat}^o \rightarrow \text{Cat}^{op} \) is a functor which assigns to each object \((X, M)\) of \( \text{Cat}^o \) the object \( X \) and to every morphism \((f, \xi): (X, M) \rightarrow (Y, L)\) the corresponding morphism \( X \xrightarrow{f} Y \) in \( \text{Cat}^{op} \) is a fibered category with the fiber at \( X \in \text{ObCat}^{op} \) equal to \( C^o_X \). This fibered category corresponds to the identical functor \( \text{Cat} = (\text{Cat}^{op})^{op} \rightarrow \text{Cat} \).

2.3.2. The fibered category \( \text{Cat}^o_{\mathcal{U}} \). Let \(|\pi|\) denote the composition of the functor \( \pi: \text{Cat}^o_{\mathcal{U}} \rightarrow \text{Cat}^{op}_{\mathcal{U}} \) and the canonical functor \( \text{Cat}^{op}_{\mathcal{U}} \rightarrow |\text{Cat}^o_{\mathcal{U}}| \). The pair \((\text{Cat}^o_{\mathcal{U}}, |\pi|)\) is a fibered category which we denote by \( \text{Cat}^o_{\mathcal{U}} \).

2.4. The universal property of the fibered category \( \text{Cat}^o_{\mathcal{U}} \). Fix two universums, \( \mathcal{U} \) and \( \mathcal{V} \) such that \( \mathcal{U} \in \mathcal{V} \). Denote by \( \text{Fib}_{\mathcal{U}, \mathcal{V}} \) the 2-category of fibered categories \( \mathcal{F} \rightarrow \mathcal{E} \) such that the 'base' \( \mathcal{E} \) belongs to \( \mathcal{V} \) and all fibers, \( \mathcal{F}_x, x \in \text{Ob} \mathcal{E}, \) belong to \( \mathcal{U} \).

Let \( \text{MFib}_{\mathcal{U}, \mathcal{V}} \) denote the 2-subcategory of the 2-category \( \text{Fib}_{\mathcal{U}, \mathcal{V}} \) formed by cartesian functors (1-morphisms) which induce category equivalence of fibers.

2.4.1. Proposition. (a) For any fibered category \( \mathcal{F} \) which belongs to \( \text{Fib}_{\mathcal{U}, \mathcal{V}} \), there exists a cartesian functor \( \psi_\mathcal{F}: \mathcal{F} \rightarrow \text{Cat}^o_{\mathcal{U}} \) such that for all cartesian functors \( F: \mathcal{F} \rightarrow \mathcal{G} \) which belong to \( \text{MFib}_{\mathcal{U}, \mathcal{V}} \), the diagram

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{F} & \mathcal{G} \\
\psi_\mathcal{F} \downarrow & & \downarrow \psi_\mathcal{G} \\
\text{Cat}^o_{\mathcal{U}} & &
\end{array}
\]

commutes in the 2-category sense.

(b) The union of images of the functors \( \psi_\mathcal{F} \) is \( \text{Cat}^o_{\mathcal{U}} \).

(c) In particular, the fibered category \( \text{Cat}^o_{\mathcal{U}} \) is a final object of \( \text{MFib}_{\mathcal{U}, \mathcal{V}} \) iff \(|\text{Cat}^o_{\mathcal{U}}|\) belongs to \( \mathcal{V} \).

Proof. Fix a pseudo-functor \( \beta: |\text{Cat}|_{\mathcal{U}} = (|\text{Cat}^o_{\mathcal{U}}|)^{op} \rightarrow \text{Cat}_{\mathcal{U}} \) quasi-inverse to the projection \( \pi: \text{Cat}^o_{\mathcal{U}} \rightarrow |\text{Cat}^o_{\mathcal{U}}| \); i.e. \( \beta \) assigns to each object \( X \) of \(|\text{Cat}^o|\) the category \( C^{op}_X \) and to any morphism \( X \xrightarrow{f} Y \) the functor \( f^{op}: C^{op}_Y \rightarrow C^{op}_X \) opposite to a (chosen) inverse image functor of \( f \). This pseudo-functor determines a fibered category \( \text{Cat}_{\beta} \) equivalent to \( \text{Cat}^o_{\mathcal{U}} \). Recall that objects of \( \text{Cat}_{\beta} \) are pairs \((X, x)\), where \( X \in \text{ObB}, x \in \text{Ob} \text{Cat}_X \), and morphisms \((X, x) \rightarrow (Y, y)\) are pairs \((f, \xi)\), where \( f \) is a morphism \( X \rightarrow Y \) and \( \xi \) is a morphism \( x \rightarrow f^*(y) \). Projection \( p_\beta: \mathcal{F}_\beta \rightarrow \text{B} \) maps \((X, x)\) to \( X \) and \((f, \xi)\) to \( f \).

For any fibered category \( \mathcal{F} = (\mathcal{F} \xrightarrow{p} \text{B}) \), there exists a natural functor \( \Psi_{\mathcal{F}}: \text{B} \rightarrow |\text{Cat}^o| \) which sends any object \( X \) of \( \text{B} \) to the class, \(|\text{Fib}_{\mathcal{F}}^o|\), of the fiber \( \mathcal{F}_X \) over \( X \) and any morphism \( X \xrightarrow{f} Y \) to the morphism \(|\text{Fib}_{\mathcal{F}}^o| \rightarrow |\text{Fib}_{\mathcal{F}}^Y|\) of \(|\text{Cat}^o|\). The composition of \( \Psi_{\mathcal{F}}: \text{B} \rightarrow |\text{Cat}|_{\mathcal{U}} \) with the pseudo-functor \( \beta: |\text{Cat}|_{\mathcal{U}} \rightarrow \text{Cat}_{\mathcal{U}} \) is a pseudo-functor, \( \beta_{\mathcal{F}} \), quasi-inverse to \( p: \mathcal{F} \rightarrow \text{B} \). The fibered category \( \mathcal{F} \) is equivalent to the fibered category...
\[ \mathfrak{F}_\beta = (\mathcal{F}_\beta \xrightarrow{\rho_\beta} \mathcal{B}) \]. It follows from the construction that the functor \( \Psi : \mathcal{B} \longrightarrow |\text{Cat}|^\circ \) defined above gives rise to a functor \( \tilde{\Psi} : \mathcal{F}_\beta \longrightarrow \text{Cat}_U \) such that the pair \((\tilde{\Psi}, \Psi)\) is a cartesian functor from \( \mathcal{F}_\beta \xrightarrow{\rho} \mathcal{B} \) to \( \text{Cat}_U^\circ \). Taking the composition with the equivalence \( \mathfrak{F} \xrightarrow{\rho} \mathcal{F}_\beta \), we obtain a cartesian functor \( \psi_\beta : \mathfrak{F} \longrightarrow \text{Cat}_U \).

Let \( \mathfrak{G} = (\mathcal{G} \xrightarrow{q} \mathcal{D}) \) be another fibered category. It follows from the construction of functors \( \psi_\beta \) that for any cartesian functor \( F : \mathfrak{F} \longrightarrow \mathfrak{G} \), the diagram

\[
\begin{array}{ccc}
\mathfrak{F} & \xrightarrow{F} & \mathfrak{G} \\
\psi_\beta \downarrow & & \downarrow \psi_\mathfrak{G} \\
\text{Cat}_2 & & \\
\end{array}
\]

which commutes in the 2-category sense.

2.5. The universal property of the fibered category \((\text{Cat}^\circ, \pi)\). As in 2.4, fix universums \( \mathfrak{U} \) and \( \mathfrak{U} \) such that \( \mathfrak{U} \in \mathfrak{V} \). Recall that a fibered category over \( \mathcal{E} \) is split (scindée) if it corresponds to a functor \( \mathcal{E}^{op} \longrightarrow \text{Cat} \) called splitting (scindage). Consider the category \( \mathcal{S}_\text{scind}_{\mathfrak{U}, \mathfrak{V}} \) formed by split fibered categories \( \mathcal{F} \rightleftarrows \mathcal{E} \) such that \( \mathcal{E} \) belongs to the universum \( \mathfrak{V} \) with all fibers from the universum \( \mathfrak{U} \). Morphisms are functors preserving splittings as 1-morphisms. The split category \( \text{Cat}_{\mathfrak{U}} \) is a final object of the 2-category \( \mathcal{S}_\text{scind}_{\mathfrak{U}, \mathfrak{V}} \) the same sense as \( \text{Cat}_{2, \mathfrak{U}} \) is a final object of the 2-category \( \text{Fib}_{\mathfrak{U}, \mathfrak{V}} \) (see 2.4.1).

2.6. Quasi-coherent modules. The material of this subsection is borrowed from [KR3]. We refer to [KR3] for more detail and proofs.

2.6.1. Modules and quasi-coherent modules on a category over a category. Let \( \mathcal{E} \) be a category (which belongs to some universum \( \mathfrak{U} \)) and \( \mathfrak{F} = (\mathcal{F}, \mathcal{F} \xrightarrow{\pi} \mathcal{E}) \) a category over \( \mathcal{E} \). Denote by \( \text{Mod}(\mathfrak{F}) \) the category opposite to the category of all sections of \( \mathfrak{F} \). We shall call objects of \( \text{Mod}(\mathfrak{F}) \) modules on \( \mathfrak{F} \).

We denote by \( \text{Qcoh}(\mathfrak{F}) \) the category opposite to the category \( \text{Cart}_{\mathcal{E}}(\mathfrak{F}, \mathfrak{F}) \) of cartesian sections of \( \mathfrak{F} \). In other words, \( \text{Qcoh}(\mathfrak{F}) = (\text{Lim}\mathfrak{F})^{op} \) (cf. [KR3], A3.5.5). Objects of \( \text{Qcoh}(\mathfrak{F}) \) will be called quasi-coherent modules on \( \mathfrak{F} \).

Any morphism \( \mathfrak{F} \longrightarrow \mathfrak{G} \) of \( \mathcal{E} \)-categories induces a functor \( \text{Mod}(\mathfrak{F}) \longrightarrow \text{Mod}(\mathfrak{G}) \). Thus, we have a functor \( \text{Mod} : \text{Cat}/\mathcal{E} \longrightarrow \text{Cat} \) from the category of \( \mathcal{E} \)-categories to the category of categories.

Similarly, the map \( \mathfrak{F} \longrightarrow \text{Qcoh}(\mathfrak{F}) \) extends to a functor \( \text{Qcoh} : \text{Cart}_{\mathcal{E}} \longrightarrow \text{Cat} \) from the category of cartesian functors over \( \mathcal{E} \) to \( \text{Cat} \).

Let \( \mathfrak{F} \) be a fibered category corresponding to a pseudo-functor \( \mathcal{E}^{op} \longrightarrow \text{Cat} \),

\[
\text{Ob}\mathcal{E} \ni X \hookrightarrow \mathcal{F}_X, \; \text{Hom}\mathcal{E} \ni f \hookrightarrow f^*, \; \text{Hom}\mathcal{E} \times_{\text{Ob}\mathcal{E}} \text{Hom}\mathcal{E} \ni (f, g) \mapsto c_{f,g} \quad (1)
\]

(cf. [KR3], A3.7, A3.7.1). Then the category \( \text{Mod}(\mathfrak{F}) \) can be described as follows. An object of \( \text{Mod}(\mathfrak{F}) \) is a function which assigns to each \( T \in \text{Ob}\mathcal{E} \) an object \( M(T) \) of the fiber \( \mathcal{F}_T \) and to each morphism \( f : T \longrightarrow T' \) a morphism \( \xi_f : f^*(M(T')) \longrightarrow M(T) \) such that \( \xi_{gf} \circ c_{f,g} = \xi_f \circ f^*(\xi_g) \). Morphisms are defined in a natural way.

An object \((M, \xi)\) of \( \text{Mod}(\mathfrak{F}) \) belongs to the subcategory \( \text{Qcoh}(\mathfrak{F}) \) iff \( \xi_f \) is an isomorphism for all \( f \in \text{Hom}\mathcal{E} \).
2.6.2. The pseudo-functor $Qcoh$. Let $\mathcal{U}$, $\mathcal{V}$ be two universums such that $\mathcal{U} \in \mathcal{V}$. Denote by $\Cart_{\mathcal{U}, \mathcal{V}}$ the 2-category whose objects are categories over categories $\mathfrak{F} = (F \to \mathcal{E})$ such that the base $\mathcal{E}$ belongs to $\mathcal{V}$ and each fiber belongs to $\mathcal{U}$. Denote by $\mathcal{M}Cart_{\mathcal{U}, \mathcal{V}}$ all cartesian functors (1-morphisms of $\Cart_{\mathcal{U}, \mathcal{V}}$)

\[
\begin{array}{ccc}
\mathcal{F}' & \xrightarrow{u} & \mathcal{F} \\
\pi' \downarrow & & \downarrow \pi \\
\mathcal{E}' & \xrightarrow{v} & \mathcal{E}
\end{array}
\]

such that the functors induced on fibers are category equivalences. It follows that $\mathcal{M}Cart_{\mathcal{U}, \mathcal{V}}$ is a 2-subcategory of the 2-category $\Cart_{\mathcal{U}, \mathcal{V}}$ (and the 2-category $\Fib_{\mathcal{U}, \mathcal{V}}$ introduced in 2.4 is a full 2-subcategory of $\mathcal{M}Cart_{\mathcal{U}, \mathcal{V}}$).

2.6.2.1. Proposition. The map $\mathfrak{F} \mapsto Qcoh(\mathfrak{F})$ extends to a pseudo-functor

$$Qcoh : \mathcal{M}Cart_{\mathcal{U}, \mathcal{V}}^{\text{op}} \to \mathcal{V}.$$

Proof. See [KR3, 11.1.4.2].

The pseudo-functor $Qcoh$ gives rise to a functor $\mathfrak{Sp} : \mathcal{M}Cart_{\mathcal{U}, \mathcal{V}} \to |\mathcal{Cat}|_{\mathcal{V}}$. Thus by definition $C_{\mathfrak{Sp}(\mathfrak{F})} = Qcoh(\mathfrak{F})$ for any category $\mathfrak{F}$ over $\mathcal{E}$. We call $\mathfrak{Sp}(\mathfrak{F})$ the categoric spectrum of $\mathfrak{F}$.

2.6.3. Quasi-coherent modules on presheaves of sets. Let $X$ be a presheaf of sets on the base $\mathcal{E}$, i.e. a functor $\mathcal{E}^{\text{op}} \to \mathcal{Set}$. Then we have a functor $\mathcal{E}/X \to \mathcal{E}$ and the category $\mathfrak{F}/X = \mathfrak{F} \times_\mathcal{E} \mathcal{E}/X$ over $\mathcal{E}/X$ obtained via a base change (as usual, we identify $\mathcal{E}$ with a full subcategory of the category $\mathcal{E}^{\text{op}}$ of presheaves of sets on $\mathcal{E}$ formed by representable presheaves). Notice that any morphism of the category $\mathcal{E}/X$ over $\mathcal{E}$ is cartesian. Therefore, by [KR3, 11.1.4.1], the category $Qcoh(\mathfrak{F}/X)$ is equivalent to the category $\mathcal{C}art_\mathcal{E}(\mathcal{E}/X, \mathfrak{F})^{\text{op}}$ opposite to the category of cartesian functors $\mathcal{E}/X \to \mathfrak{F}$.

3. Certain classes of morphisms in $|\mathcal{Cat}|^\alpha$ and $\mathcal{Cat}^{\text{op}}$.

3.1. Left exact, right exact, and exact morphisms. A morphism $X \xrightarrow{f} Y$ is called right exact (resp. left exact, resp. exact), if its inverse image functor preserves colimits (resp. limits, resp. both limits and colimits) of arbitrary finite diagrams.

The following assertion is a reformulation of Proposition 1.1.4 in [GZ].

3.1.1. Proposition. Let $f = p_f \circ f_c$ be the canonical decomposition of a morphism $X \xrightarrow{f} Y$ into a conservative morphism $X \xrightarrow{f_c} \Sigma^{-1}Y$ and a localization $\Sigma^{-1}Y \xrightarrow{p_f} Y$. Suppose $C_Y$ has finite limits (resp. finite colimits). Then $f$ is left exact (resp. right exact) iff the class of arrows $\Sigma_f$ satisfies left (resp. right) Ore conditions. In this case both the localization $p_f$ and the conservative component $f_c$ are left (resp. right) exact.

In particular, if the category $C_Y$ has limits and colimits of finite diagrams, then $f$ is exact iff both the localization $p_f$ and the conservative component $f_c$ are exact. The exactness of $p_f$ is equivalent to that $\Sigma_f$ satisfies left and right Ore conditions.
3.2. Dualization functor and dual notions. The dualization functor

\[ o : \mathcal{C} \to \mathcal{C} \]

assigns to each object \( Y \) of \( \mathcal{C} \) the object \( Y^o \) defined by \( C_{Y^o} = C_Y^{op} \), and to each morphism \( f \) with an inverse image functor \( f^* \), the morphism \( f^o \) having \((f^*)^{op}\) as an inverse image functor. It follows that the dualization functors is an automorphism of the category \( \mathcal{C} \) and its square is the identical functor.

The dualization functor maps left (resp. right) exact morphisms to right (resp. left) exact morphisms. Conservative morphisms and localizations are stable under the dualization. In particular, the dualization functor preserves the canonical decomposition:

\[ f = p_f^o \circ f^o \circ f_c^o \]

3.3. Continuous and cocontinuous morphisms. Duality. A morphism \( X \to Y \) in \( \mathcal{C} \) is called cocontinuous if \( f^o \) is continuous. In other words, \( f \) is cocontinuous iff its inverse image functor, \( f^* \), has a left adjoint, \( f^! \).

Denote by \( \mathcal{C}_c \) the subcategory of \( \mathcal{C} \) formed by continuous morphisms and by \( \mathcal{C}_{coc} \) the subcategory of \( \mathcal{C} \) formed by cocontinuous morphisms. The dualization functor induces isomorphisms of categories \( \mathcal{C}_c \to \mathcal{C}_{coc} \)

The map which assigns to every continuous morphism \( X \to Y \) with a direct image functor \( f_* \) the morphism \( X \to Y \) having \( f_* \) as an inverse image functor, defines an isomorphism of categories

\[ \wedge : \mathcal{C}_c = (\mathcal{C}_{coc}^{op}) \to \mathcal{C}_{coc}^{op} \]

The inverse isomorphism,

\[ \vee : \mathcal{C}_{coc} = (\mathcal{C}_c^{op}) \to \mathcal{C}_c^{op} \]

assigns to any cocontinuous morphism \( X \to Y \) with an inverse image functor \( g^* \) the morphism \( Y \to X \) having as an inverse image functor a left adjoint, \( g^! \), to \( g^* \).

The composition

\[ \mathcal{D} \wedge : \mathcal{C}_c \to \mathcal{C}_c^{op} \]

of the functor \( \wedge \) with the dualization functor \( o \) is a duality on \( \mathcal{C}_c^{op} \), i.e. a contravariant functor whose square is the identical functor.

Similarly, the composition

\[ \mathcal{D} \vee : \mathcal{C}_{coc} \to \mathcal{C}_{coc}^{op} \]

of the functor \( \vee \) with the dualization functor \( o \) is a duality on \( \mathcal{C}_{coc}^{op} \).

3.4. Flat and coflat morphisms. We call a morphism \( f \to Y \) flat if it is continuous and its inverse image functor is exact. Since \( f^* \) preserves colimits of arbitrary small
diagrams, the exactness requirement means that \( f^* \) preserves limits of finite diagrams, i.e. \( f \) is left exact (see (ii) above).

By 3.1.1, a continuous morphism \( f \) is flat iff both the localization \( p_f \) and the conservative morphism \( f_c \) in the decomposition \( f = p_f \circ f_c \) are exact. One can show that both \( p_f \) and \( f_c \) are continuous, hence flat (see [R3]).

3.4.1. Coflat morphisms. A continuous morphism \( f \) is called coflat if the dual morphism \( \mathcal{D}^\wedge(f) \) is flat; i.e. \( f \) is coflat iff its direct image, \( f_* \), is exact, or, equivalently, \( f_* \) preserves colimits of finite diagrams.

3.4.2. Weakly flat and coflat morphisms. A pair of arrows \( g_1, g_2 : M \rightarrow L \) is called coreflexive, if there exists a morphism \( h : L \rightarrow M \) such that \( h \circ g_1 = id_M = h \circ g_2 \).

Dually, a pair of arrows \( g_1, g_2 : M \rightarrow L \) is called reflexive, if there exists a morphism \( h : L \rightarrow M \) such that \( g_1 \circ h = id_M = g_2 \circ h \).

We call a continuous morphism \( X \xrightarrow{f} Y \) in \( |\mathcal{Cat}|^o \) weakly flat if its inverse image functor preserves kernels of coreflexive pairs of arrows.

Dually, \( f \) is called weakly coflat if its direct image functor preserves cokernels of reflective pairs of arrows.

3.5. Affine morphisms. A continuous morphism \( X \xrightarrow{f} Y \) is called affine if its direct image functor \( f_* \) has a right adjoint, \( f^! \), and is conservative.

Applying the duality \( \mathcal{D}^\wedge \), one can identify an affine morphism \( X \xrightarrow{f} Y \) with a conservative morphism \( Y \rightarrow X \) with an inverse image functor \( f^* : X \rightarrow Y \) which is both continuous and cocontinuous.

3.5.1. Coaffine morphisms. We call a continuous morphism \( X \xrightarrow{f} Y \) coaffine iff the dual morphism \( \mathcal{D}^\wedge(f) \) is affine. In other words, \( f \) is coaffine iff its inverse image functor \( f^* \) is conservative and has a left adjoint, \( f_! \).

3.6. Classes of morphisms of \( \mathcal{C}at^{op} \). Given a class \( \mathcal{M} \) of morphisms of \( |\mathcal{Cat}|^o \), we say that a 1-morphism in \( \mathcal{C}at^{op} \) belongs to \( \mathcal{M} \) if its image by the canonical functor \( \mathcal{C}at^{op} \rightarrow |\mathcal{Cat}|^o \) belongs to \( \mathcal{M} \). Thus we have continuous, flat, affine etc. 1-morphisms of the category \( \mathcal{C}at^{op} \).

3.7. Example: canonical morphisms of the category \( |\mathcal{Cat}|^o \). Let \( \bullet \) be the initial object of the category \( |\mathcal{Cat}|^o \) such that \( C_\bullet \) is the category with one (hence identical) morphism. For any object \( X \) of \( |\mathcal{Cat}|^o \), denote by \( f_X \) the unique morphism \( \bullet \rightarrow X \).

3.7.1. Lemma. The unique inverse image functor \( f_X^* : C_X \rightarrow C_\bullet \) has a right adjoint, \( f_{X*} \), (i.e. \( f_X \) is continuous) iff the category \( C_X \) has a final object. In this case, a direct image functor \( f_{X*} \) maps the unique object of the category \( C_\bullet \) to a final object of the category \( C_X \).

Dually, the functor \( f_X^* \) has a left adjoint, \( f_{X!} \) (i.e. \( f_X \) is cocontinuous), iff the category \( C_X \) has an initial object.

Proof is left to the reader.
3.7.2. Further observations. All functors from $C_\bullet$, in particular $f_\bullet$, are conservative. Thus $f_\bullet: \bullet \to X$ is affine iff there exists a direct image functor $f_\bullet^*$ (i.e. iff the category $C_X$ has a final object), and the functor $f_\bullet^*$ has a right adjoint, $f_\bullet^!$.

Since there is only one functor $C_X \to C_\bullet$, the functor $f_\bullet^*$ has a right adjoint iff it is left adjoint to the functor $f_\bullet!$. In this case, $f_\bullet^*$ is isomorphic to $f_\bullet^!$, hence it maps the unique object of $C_\bullet$ to an initial object of $C_X$.

Thus, the unique morphism $\bullet \to X$ is affine iff the category $C_X$ has a zero object, i.e. final objects in $C_X$ are initial too.

If the category $C_X$ has an initial object, then there is a continuous morphism $\phi_X: X \to \bullet$ whose inverse image functor maps the unique object of $\bullet$ to an initial object of $C_X$. The direct image functor of $\phi_X$ is the unique functor $\phi_*: C_X \to C_\bullet$ (which coincides with the functor $f_\bullet^!$ above).

The functor $f_\bullet^*$ has a right adjoint, $\phi_X^!$, iff the category $C_X$ is marked. Notice, however, that the morphism $\phi_X$ is affine (that is the functor $\phi_X^*$ is conservative) iff all arrows of $C_X$ are invertible, i.e. $C_X$ is a connected groupoid, or, what is the same, $\phi_X^*$ is a category equivalence, i.e. $X \simeq \bullet$.

3.8. Example: cocontinuous and coaffine morphisms between spectra of rings. Let $R$ and $S$ be associative unital rings. A morphism $\text{Sp}(S) \to \text{Sp}(R)$ is cocontinuous iff its inverse image functor is isomorphic to

$$\text{Hom}_R(\mathcal{N},-): R \text{-mod} \to S \text{-mod}, \quad L \mapsto \text{Hom}_R(\mathcal{N},L).$$

for some $(R,S)$-bimodule $\mathcal{N}$.

The functor (1) has a left adjoint, $f_!$, iff it is isomorphic to the functor (3) for some $(R,S)$-bimodule $\mathcal{N}$; or, equivalently, the canonical functorial morphism

$$\mathcal{M} \otimes_R L \to \text{Hom}_R(\mathcal{M}_R, L)$$

is an isomorphism for all $L$ (equivalently, (4) is an isomorphism for $L = \mathcal{M}_R$). Here $\mathcal{M}_R = \text{Hom}_R(\mathcal{M}, R)$. This happens iff $\mathcal{M}$ is a projective $R$-module of a finite type.

Thus, affine morphisms $\text{Sp}(R) \to \text{Sp}(S)$ are in bijective correspondence with isomorphism classes of $(S,R)$-bimodules $\mathcal{M}$ which are strictly projective as left $S$-modules. Recall that strictly projective means projective cogenerator of finite type.

Coaffine morphisms $\text{Sp}(R) \to \text{Sp}(S)$ are in bijective correspondence with isomorphism classes of $(S,R)$-bimodules $\mathcal{M}$ which are strictly projective as right $R$-modules.

4. Continuous morphisms to the categoric spectrum of a ring and 'structure sheaves’. Let $R$ be an associative unital ring. For a morphism $X \to \text{Sp}(R)$ with an inverse image functor $f^*$, we denote by $\mathcal{O}$ the object $f^*(R)$. It follows that the object $\mathcal{O}$ is determined by the pair $(X,f)$ uniquely up to isomorphism. The functor $f^*$ defines a monoid morphism $\text{End}_R(R) \to C_X(\mathcal{O},\mathcal{O})$ the whose composition with the canonical ring isomorphism $R^\circ \to \text{End}_R(R)$ gives a monoid morphism $\phi_f: R \to C_X(\mathcal{O},\mathcal{O})^\circ$. Here $C_X(\mathcal{O},\mathcal{O})^\circ$ denotes the monoid opposite to $C_X(\mathcal{O},\mathcal{O})$. If the category $C_X$ is preadditive and the functor $f^*$ is additive, the morphism $\phi_f$ is a unital ring morphism.
In general, the object $O$ does not determine the morphism $f$. It does, however, if $f$ is continuous:

4.1. Proposition. Let $X \xrightarrow{f} \text{Sp}(R)$ be a continuous morphism. Then
(a) The morphism $f$ is determined by $O = f^*(R)$ uniquely up to isomorphism.
(b) There exists a coproduct of any small set of copies of $O$.
(c) The object $O$ has a structure of an $R$-module in the category $C_X^{op}$. In particular, $O$ is an abelian cogroup in the category $C_X$ (i.e. an abelian group in $C_X^{op}$) and the canonical map $\phi_f : R \to C_X(O, O)^\circ$ is a ring morphism.

Proof. (a) Let $f_*$ be a direct image functor of $f$ (i.e. a right adjoint to $f^*$). We have functorial isomorphisms $C_X(f^*(R), M) \simeq \text{Hom}_R(R, f_*(M)) \simeq f_*(M)$ which shows that the direct image functor $f_*$ of $f$ is naturally isomorphic to the functor $M \mapsto C_X(f^*(R), M)$. Therefore the inverse image functor $f^*$ of $f$ is defined uniquely up to isomorphism (being a left adjoint to $f_*$) by the object $f^*(R)$. Since $f^*$ preserves colimits, there exists a coproduct of any set of copies of $O(X, f) = f^*(R)$.

(b) Since $f^*$ preserves colimits, there exists a coproduct of any set of copies of $O(X, f) = f^*(R)$.

(c) The assertion follows from the isomorphism $f_* \simeq C_X(O, -)$ and the fact that $f_*$ takes values in the category of $R$-modules.

4.1.1. Global sections functor. The object $O$ is viewed as the 'structure sheaf' on $X$. We denote $C_X(O, O)^\circ$ by $\Gamma_X O$. The functor

$$f_{\circ*} : C_X \longrightarrow \Gamma_X O - \text{mod}, \quad M \mapsto C_X(O, M)$$

will be called the 'global sections functor' on $(X, O)$. In particularly, $\Gamma_X O = f_{\circ*}(O)$ is the ring of global sections of the structure sheaf $O$.

4.2. Right $R$-modules. We call $R$-modules in $C_X^{op}$ right $R$-modules in $C_X$, or right $R$-modules on $X$. Right $R$-modules on $X$ form a category which we denote by $R^\circ - \text{Mod}_X$. A morphism from an $R^\circ$-module $(M, \phi)$ to an $R^\circ$-module $(M', \phi')$ is a morphism $h : M \to M'$ of (abelian) groups in the category $C_X$ such that the diagram

\[
\begin{array}{ccc}
C_X(M, M) & \xrightarrow{C_X(M, h)} & C_X(M, M') \\
\phi \uparrow & & \uparrow C_X(h, M) \\
R & \xrightarrow{\phi'} & C_X(M', M')
\end{array}
\] (1)

commutes. The composition is defined in an obvious way.

4.3. Proposition. For every continuous morphism $X \xrightarrow{f} Y$, its inverse image functor induces a functor $R^\circ - \text{Mod}_Y \longrightarrow R^\circ - \text{Mod}_X$.

Proof. In fact, the functorial isomorphism $C_X(f^*(M), T) \simeq C_Y(M, f_*(T))$ implies that if $M$ has a structure of an abelian group in the category $C_Y^{op} = C_Y^{op}$ (i.e. the functor $C_Y(M, -) : C_Y^{op} \longrightarrow \text{Sets}$, has a lifting to a functor $C_Y^{op} \longrightarrow \mathbb{Z} - \text{mod}$), then $f^*(M)$
has a structure of an abelian group in the category $C_X^{op} = C_{X^*}$. In particular, for any abelian group $M$ in $C_Y^{op}$ and for any object $T$ in $C_Y$, the morphism $f_{M,T}^* : C_Y(M,T) \to C_X(f^*(M), f^*(T)) \simeq C_Y(M, f_* f^*(T))$ is an abelian group morphism. It follows that $f_{M,M}^* : C_Y(M,M) \to C_X(f^*(M), f^*(M))$ is a ring morphism. Thus if $R^o \to C_Y(M,M)$ is a ring morphism (i.e. a right $R$-module structure on $M$), its composition with $f_{M,M}^*$ is a right $R$-module structure on $f^*(M)$. This extends to a functor $R^o \to Mod_Y \to R^o \to Mod_X$, hence the assertion.

Thus, for any associative ring $R$, we have a pseudo-functor

$$\beta_R : |Cat| = (|Cat|^o)^{op} \to \mathbf{Cat}$$

which defines the fibered category $(R^o \to Mod \to |Cat|^o)$. Here $|Cat|^o$ is the subcategory of the category $|Cat|^o$ formed by the continuous morphisms.

### 4.4. Right $R$-modules and continuous morphisms to $Sp(R)$: functorial picture

We denote by $|Cat|^o(R)$ the category of right $R$-modules. Its objects are triples $(X, \mathcal{O}_X, \phi)$, where $X$ is an object of $|Cat|^o$, and $(\mathcal{O}_X, \phi)$ is a right $R$-module in the category $C_X$ (cf. 4.2). Morphisms from $(X, \mathcal{O}_X, \phi)$ to $(Y, \mathcal{O}_Y, \psi)$ aremorphisms $X \to Y$ such that there exists an isomorphism $\lambda : f^*(O_Y) \sim \mathcal{O}_X$ making the diagram

$$\begin{array}{ccc}
R^o & \xrightarrow{\phi} & C_X(\mathcal{O}_X, \mathcal{O}_X) \\
\downarrow{\psi} & & \downarrow{c_\lambda} \\
C_Y(\mathcal{O}_Y, \mathcal{O}_Y) & \xrightarrow{f_{\mathcal{O}_Y,\mathcal{O}_Y}^*} & C_X(f^*(\mathcal{O}_Y), f^*(\mathcal{O}_Y))
\end{array}$$

commute. Here $c_\lambda$ denotes the conjugation by $\lambda$, $h \mapsto \lambda^{-1} \circ h \circ \lambda$.

Let $Cat^{op}(R)$ denote the category whose objects are same as objects of $|Cat|^o(R)$. Morphisms from $(X, \mathcal{O}_X, \phi)$ to $(Y, \mathcal{O}_Y, \psi)$ are given by pairs $(f^*, \lambda)$, where $f^*$ is a functor $C_Y \to C_X$, $\lambda$ an isomorphism $f^*(\mathcal{O}_Y) \sim \mathcal{O}_X$ such that the diagram (2) commutes. Composition is defined in an obvious way. The canonical functor $Cat^{op}(R) \to |Cat|^o(R)$ is a fibered category.

**4.4.1. Note.** Morphisms of the categories $Cat^{op}(R)$ and $|Cat|^o(R)$ (more precisely, the meaning of the component $\lambda$ of a morphism in $Cat^{op}(R)$, see above) might be viewed as follows. For any object $X$ of $|Cat|^o$, let $Cat^{op}(R)_X$ denote the fiber of the category $Cat^{op}(R)$ over $X$; its objects are triples $(X, \mathcal{O}_X, \phi)$, where $\phi$ is a right action of the ring $R$ on $\mathcal{O}_X$, morphisms are morphisms of actions. An inverse image functor, $f^*$, of a morphism $X \to Y$ induces a functor

$$f_R^* : Cat^{op}(R)_Y \to Cat^{op}(R)_X$$

which maps an object $(Y, \mathcal{O}_Y, \psi)$ to the object $(X, f^*(\mathcal{O}_Y), \psi f^*)$, where the action $\psi f^*$ is the composition of $\psi : R^o \to C_Y(\mathcal{O}_Y, \mathcal{O}_Y)$ and the ring morphism

$$f_{\mathcal{O}_Y,\mathcal{O}_Y}^* : C_Y(\mathcal{O}_Y, \mathcal{O}_Y) \to C_X(f^*(\mathcal{O}_Y), f^*(\mathcal{O}_Y)).$$
The commutativity of the diagram (2) means exactly that \( \lambda : f^*(\mathcal{O}_Y) \rightarrow \mathcal{O}_X \) is an isomorphism \( f_R^*(Y, \mathcal{O}_Y, \psi) \rightarrow (X, \mathcal{O}_X, \phi) \).

We denote by \( |\text{Cat}^o(R)|_e \) the full subcategory of the category \( |\text{Cat}|^o(R) \) whose objects are triples \((X, \mathcal{O}_X, \phi)\) such that the following conditions hold:

(a) \((\mathcal{O}_X, \phi)\) is a right \( R \)-module in the category \( \text{C}_X \), i.e. \( \mathcal{O}_X \) is an abelian group in \( C_X^{op} \) and \( \phi \) a ring morphism \( R^o \rightarrow C_X(\mathcal{O}_X, \mathcal{O}_X) \) (cf. 4.1(c) and 4.2).

(b) There exists a coproduct of any set of copies of \( \mathcal{O}_X \).

(c) Let \( \Phi_\phi \) denote the functor from the subcategory \( \mathcal{L}_R \) of free \( R \)-modules to \( \text{C}_X \) which is uniquely defined by the action \( \phi \) (thanks to (b) above). The image by \( \Phi_\phi \) of any pair of arrows \( X_1 \Rightarrow X_0 \) has a cokernel.

Let \( \text{Cat}^{op}(R)_e \) denote the preimage of \( |\text{Cat}^o(R)|_e \) in \( \text{Cat}^{op}(R) \). On the other hand, let \( (|\text{Cat}^o/\text{Sp}(R))_e \) denote the full subcategory of the category \( |\text{Cat}^o/\text{Sp}(R) \) whose objects are continuous morphisms to \( \text{Sp}(R) \), and let \( (\text{Cat}^{op}/\text{Sp}(R))_e \) denote its preimage in \( \text{Cat}^{op}/\text{Sp}(R) \).

**4.4.2. Proposition.** The functor (3) induces an equivalence of the fibered categories

\[
\begin{pmatrix}
|\text{Cat}^{op}/\text{Sp}(R))_e \rightarrow \text{Cat}^{op}(R)_e \\
|\text{Cat}^o(\text{Sp}(R))_e \rightarrow |\text{Cat}^o(R)_e
\end{pmatrix}
\]

**Proof.** By 4.1, the cartesian morphism (3) induces a cartesian morphism (4).

Thanks to 4.1, in order to prove that (4) is an equivalence, it suffices to show that the functor \( f^* \) can be reconstructed (uniquely up to isomorphism) from the right \( R \)-module \((\mathcal{O}, R^f, C_X(\mathcal{O}, \mathcal{O})^o)\) associated with \( f \). In fact, the right \( R \)-module \((\mathcal{O}, \phi_f)\) gives rise to a functor, \( \Phi_f \), from the category \( \mathcal{L}_R \) of free left \( R \)-modules (a full subcategory of \( R^{mod} \)) to \( C_X \) which is isomorphic to the restriction of \( f^* \) to \( \mathcal{L}_R \): direct sums of copies of \( R \) are mapped to direct sums of copies of \( \mathcal{O} \) and \( \phi_f \) (regarded as \( \text{End}_R(R) \rightarrow C_X(\mathcal{O}, \mathcal{O}) \)) determines map on morphisms. Every \( R \)-module \( M \) is a cokernel of a pair of morphisms \( L_1 \Rightarrow L_0 \), where \( L_1, L_0 \) are free \( R \)-modules. Since the functor \( f^* \) preserves cokernels of pairs of arrows, \( f^*(M) \) is isomorphic to the cokernel of the pair \( \Phi_f(L_1) \Rightarrow \Phi_f(L_0) \).

**4.5. \( \mathbb{Z} \)-spaces.** Let \( X \Rightarrow \text{Sp}(R) \) be a continuous morphism with an inverse image functor \( f^* \), and let \( \mathcal{O} = f^*(R) \). It follows from 4.1(c) that the functor \( f_* = C_X(\mathcal{O}, -) \) is naturally decomposed into

\[
\begin{array}{c}
C_X \xrightarrow{f_{\mathcal{O}*}} \Gamma_X \mathcal{O}^{mod} \\
\downarrow \phi_{f*} \\
R^{mod}
\end{array}
\]

(5)

where \( \phi_{f*} \) is the pull-back by the ring morphism \( \phi_f : R \rightarrow \Gamma_X \mathcal{O} \) defining a right \( R \)-module structure on \( \mathcal{O} \).

**4.5.1. Lemma.** The global sections functor \( f_{\mathcal{O}*} \) is a direct image functor of a continuous morphism, \( f_{\mathcal{O}} : X \rightarrow \text{Sp}(\Gamma_X \mathcal{O}) \), if any pair of arrows \( \mathcal{O}^{\oplus I} \rightarrow \mathcal{O}^{\oplus J} \) between coproducts of copies of \( \mathcal{O} \) has a cokernel in \( C_X \).
Proof. The fact follows from (the argument of) 4.4.2 applied to the case when \( R = \Gamma_X O \): the inverse image functor \( f_o^* \) assigns to a free \( \Gamma_X O \)-module \( \Gamma_X O^{\oplus J} \) the coproduct \( O^{\oplus J} \) of \( J \) copies of the object \( O \). □

4.5.2. The category of \( \mathbb{Z} \)-spaces. Denote by \( |\text{Cat}|_O^\mathbb{Z} \) the category whose objects are all pairs \((X, \mathcal{O})\), where \( X \) is an object of \( |\text{Cat}|_O^o \) and \( \mathcal{O} \) is an abelian group in \( C_X^{op} \) such that there exist coproducts of small sets of copies of \( \mathcal{O} \) and any pair of arrows \( \mathcal{O}^{\oplus I} \rightarrow \mathcal{O}^{\oplus J} \) between coproducts of copies of \( \mathcal{O} \) has a cokernel in \( C_X \). Morphisms from \((X, \mathcal{O})\) to \((X', \mathcal{O}')\) are morphisms \( X \xrightarrow{f} X' \) such that there exists an isomorphism \( f^*(\mathcal{O}') \xrightarrow{\sim} \mathcal{O} \). Composition is defined in an obvious way. Objects of the category \( |\text{Cat}|_O^\mathbb{Z} \) will be called \( \mathbb{Z} \)-‘spaces’.

4.5.2.1. A reformulation. By 4.5.1, \( \mathbb{Z} \)-spaces are pairs \((X, \mathcal{O})\) such that \( \mathcal{O} \) is an abelian group in the category \( C_X^{op} \) and the canonical functor

\[
f_{o*} : C_X \longrightarrow \Gamma_X O - \text{mod}, \quad M \mapsto C_X(\mathcal{O}, M),
\]

has a left adjoint; or, equivalently, \( f_{o*} \) is a direct image functor of a continuous morphism.

4.5.2.2. Example. If \( C_X \) is an additive category with small coproducts and cokernels, then \((X, \mathcal{O})\) is a \( \mathbb{Z} \)-space for any \( \mathcal{O} \in \text{Ob}C_X \).

4.5.3. Affine \( \mathbb{Z} \)-spaces. We call a \( \mathbb{Z} \)-space \((X, \mathcal{O})\) affine if the canonical morphism \( f_o : X \longrightarrow \text{Sp}(\Gamma_X \mathcal{O}) \) is an isomorphism; i.e. the functor \( f_{o*} \) (see (6)) is a category equivalence. By a Mitchell’s theorem, affine \( \mathbb{Z} \)-spaces are pairs \((X, \mathcal{O})\), where \( C_X \) is an abelian category with small coproducts, and \( \mathcal{O} \) is a projective cogenerator of finite type. We denote by \( \text{Aff}_\mathbb{Z} \) the full subcategory of the category \( |\text{Cat}|_O^\mathbb{Z} \) formed by affine \( \mathbb{Z} \)-spaces.

The functor \( \text{Sp} : \text{Rings}^{op} \longrightarrow |\text{Cat}|_O^o, \quad R \longmapsto \text{Sp}(R) \), gives rise to the functor

\[
\text{Sp}_\mathbb{Z} : \text{Rings}^{op} \longrightarrow |\text{Cat}|_O^\mathbb{Z}, \quad R \longmapsto (\text{Sp}(R), R)
\]

which takes values in the subcategory \( \text{Aff}_\mathbb{Z} \). We denote the image of the functor \( \text{Sp}_\mathbb{Z} \) by \( \text{Aff}_\mathbb{Z} \). Thus, objects of the category \( \text{Aff}_\mathbb{Z} \) are pairs \((\text{Sp}(R), R)\) and morphisms from \((\text{Sp}(R), R) \rightarrow (\text{Sp}(T), T)\) are morphisms \( \text{Sp}(R) \longrightarrow \text{Sp}(T) \) corresponding to unital ring morphisms \( T \longrightarrow R \). The functor \( \text{Sp}_\mathbb{Z} \) induces an inclusion functor \( \gamma_* : \text{Aff}_\mathbb{Z} \longrightarrow |\text{Cat}|_O^\mathbb{Z} \) which takes values in the subcategory of affine \( \mathbb{Z} \)-spaces.

4.5.4. Proposition. The functor \( \gamma_* : \text{Aff}_\mathbb{Z} \longrightarrow |\text{Cat}|_O^\mathbb{Z} \) is fully faithful and has a left adjoint. In particular, the functor \( \gamma_* \) induces an equivalence of \( \text{Aff}_\mathbb{Z} \) and the category of affine \( \mathbb{Z} \)-spaces.

Proof. Let \( f \) be a morphism \((X, \mathcal{O}) \longrightarrow (X', \mathcal{O}')\). A choice of an inverse image functor, \( f^* \), of \( f \) and an isomorphism \( \lambda : f^*(\mathcal{O}') \xrightarrow{\sim} \mathcal{O} \) determines a ring morphism \( \psi_{f^*, \lambda} : \Gamma_X \mathcal{O}' \longrightarrow \Gamma_X \mathcal{O} \). One can check that the corresponding morphism of categoric spectra, \( \text{Sp}(\Gamma_X \mathcal{O}) \longrightarrow \text{Sp}(\Gamma_X \mathcal{O}') \), does not depend on choices. Thus we have a functor \( \gamma^* : |\text{Cat}|_O^\mathbb{Z} \longrightarrow \text{Aff}_\mathbb{Z} \). By 4.5.1, we have a natural morphism \( \eta_\gamma : \text{Id}_{|\text{Cat}|_O^\mathbb{Z}} \longrightarrow \gamma_* \gamma^* \). And there is an isomorphism \( \epsilon_\gamma : \gamma^* \gamma_* \xrightarrow{\sim} \text{Id}_{\text{Aff}_\mathbb{Z}} \). These are adjunction morphisms. Since \( \epsilon_\gamma \) is an isomorphism, the functor \( \gamma_* \) is fully faithful. □
4.5.5. A description of the category $\mathfrak{Aff}_{\mathbb{Z}}$. Let $\mathfrak{Ass}$ denote the category whose objects are associative rings; and morphisms from a ring $R$ to a ring $S$ are equivalence classes of ring morphisms $R \to S$ by the following equivalence relation: two ring morphisms $f, g : R \to S$ are equivalent if they are conjugated, i.e. $g(-) = tf(-)t^{-1}$ for an invertible element $t$ of $S$.

4.5.5.1. Proposition. Two ring morphisms, $R \xrightarrow{\phi} S$, are conjugate iff the corresponding inverse image functors, $R - \text{mod} \xrightarrow{\phi^*} S - \text{mod}$, are isomorphic.

Proof. (a) Suppose that $\psi$ and $\phi$ are conjugate, i.e. there exists an invertible element, $t$, of $S$ such that $\psi(r) = t\phi(r)t^{-1}$ for all $r \in R$. For any $R$-module $M = (M, m)$, we have a commutative diagram

\[
\begin{array}{c}
S \otimes M \xrightarrow{\cdot t} S \otimes M \\
\gamma_\psi \downarrow \quad \downarrow \gamma_\phi \\
S \otimes_{R, \psi} M \xrightarrow{\lambda_t} S \otimes_{R, \phi} M
\end{array}
\]

Here $\cdot t$ denotes the $S$-module morphism $s \otimes z \mapsto st \otimes z$ for all $s \in S$, $z \in M$; $\gamma_\psi$, $\gamma_\phi$ are canonical epimorphisms.

In fact, for any $s \in S$, $r \in R$, $z \in M$, $\gamma_\psi(s\phi(r) \otimes z) = \gamma_\psi(s \otimes r \cdot z)$, and $\cdot t(s \otimes r \cdot z) = st \otimes r \cdot z$.

On the other hand, $\cdot t(s\psi(r) \otimes z) = s\psi(r)t \otimes z = st\phi(r) \otimes z$, and $\gamma_\phi(st\phi(r) \otimes z) = \gamma_\phi(st \otimes r \cdot z)$. Since $\gamma_\psi$ is by definition the cokernel of two maps $\psi_l : S \otimes_k R \otimes_k M \xrightarrow{\psi_l} S \otimes_k M$, $s \otimes r \otimes z \xrightarrow{\psi_l} s\psi(r) \otimes z$, and $s \otimes r \otimes z \xrightarrow{\psi_r} s \otimes r \cdot z$,

it follows the existence of a (necessarily unique) morphism $\lambda_t : S \otimes_{R, \psi} M \to S \otimes_{R, \phi} M$ such that the diagram (1) commutes; i.e. $\lambda_t$ is given by $\gamma_\psi(s \otimes z) \mapsto \gamma_\phi(st \otimes z)$.

(b) Conversely, suppose $\phi$, $\psi$ are unital ring morphisms such that there is a functorial isomorphism $u : \psi^* \xrightarrow{\cong} \phi^*$. Identifying both $\phi^*(R)$ and $\psi^*(R)$ with the left $S$-module $S$, we obtain, in particular, an $S$-module morphism $u(R) : S \to S$. Since $S$ is a ring with unit, $u(R)$ equals to $\cdot t : s \mapsto st$ for some $t \in S$. Since $u$ is a functor morphism, for any $r \in R$, $u(R) \circ \psi^*(r) = \phi^*(r) \circ u(R)$. This means that for any $s \in S$, $s\psi(r)t = st\phi(r)$, hence $\psi(r) = t\phi(r)t^{-1}$. ■

4.5.5.2. Corollary. The functor $\mathbf{Sp}_\mathbb{Z} : \text{Rings}^{op} \to |\text{Cat}|^o_{\mathbb{Z}}$, $R \mapsto (\mathbf{Sp}(R), R)$ induces an isomorphism of categories $\mathfrak{Ass}^{op} \xrightarrow{\sim} \mathfrak{Aff}_{\mathbb{Z}}$.

4.5.5.3. Corollary. The functor $\mathbf{Sp}_\mathbb{Z} : \text{Rings}^{op} \to |\text{Cat}|^o_{\mathbb{Z}}$ induces an equivalence of categories $\mathfrak{Ass}^{op} \xrightarrow{\sim} \mathfrak{Aff}_{\mathbb{Z}}$.

Proof. This follows from 4.5.4 and 4.5.5.2. ■

4.5.6. Remark. If $S$ is a commutative ring, then for any ring $R$, the surjection $\text{Rings}(R, S) \to \mathfrak{Ass}(R, S)$ is a bijective map. In particular, the full subcategory of $\mathfrak{Ass}$
formed by commutative rings is isomorphic to the category $CRings$ of commutative rings. Thus, the equivalence of categories $Ass^{op} \rightarrow \mathbf{Aff}_Z$ induces an equivalence between the category $CRings^{op}$ opposite to the category of commutative unital rings and the full subcategory $C\mathbf{Aff}_Z$ formed by affine $\mathbb{Z}$-spaces $(X,O)$ such that the global sections ring $\Gamma_XO$ is commutative. This shows by passing that the category $C\mathbf{Aff}_Z$ of commutative affine $\mathbb{Z}$-spaces is equivalent to the category of commutative affine schemes in the usual sense.

4.6. Proposition. Let $X \in \text{Ob} \mathcal{C}_X$ be such that the category $C_X$ has cokernels of pairs of morphisms. Then $(X,O)$ is a $\mathbb{Z}$-space for any object $O$ of $C_X$ such that there exists a coproduct of any small set of copies of $O$. In particular, continuous morphisms $X \rightarrow \text{Sp}(R)$ are in one-to-one correspondence with isomorphism classes of right $R$-modules $(O,\phi)$ in $C_X$ such that $(X,O)$ is a $\mathbb{Z}$-space.

In particular, if $C_X$ is an abelian category with small coproducts, then morphisms $X \rightarrow \text{Sp}(R)$ are in one-to-one correspondence with isomorphism classes of right $R$-modules in $C_X$.

Proof. The assertion is a corollary of 4.5.4. ■

4.6.1. Example. Let $X = \text{Sp}(S)$ for some associative ring $S$, i.e. $C_X$ is the category $S-mod$ of left $S$-modules. By 4.6, continuous morphisms from $\text{Sp}(S) \rightarrow \text{Sp}(R)$ are in one-to-one correspondence with isomorphism classes of right $R$-modules in the category $S-mod$. Notice that the category of right $R$-modules in $S-mod$ is isomorphic to the category of $(S,R)$-bimodules. If $O$ is a $(S,R)$-bimodule corresponding to a morphism $f : \text{Sp}(S) \rightarrow \text{Sp}(R)$, then $L \mapsto \text{Hom}_S(O,L)$ is a direct image functor of $f$. Therefore $N \mapsto O \otimes_R N$ is an inverse image functor of $f$. Thus we have recovered a classical fact already mentioned in 3.8.1.

4.7. Additive morphisms and continuous morphisms. For any objects $X$, $Y$ of $\mathcal{C}_X^{op}$, denote by $\mathcal{C}_X^{op}(X,Y)$ the full subcategory of the category $\mathcal{C}_X^{op}(X,Y)$ whose objects are continuous morphisms. If $C_X$, $C_Y$ are (pre)additive categories, we call a morphism $X \rightarrow Y$ additive if its inverse image functor is additive. We denote by $\mathcal{C}_X^{op}(X,Y)$ the full subcategory of the category $\mathcal{C}_X^{op}(X,Y)$ whose objects are additive morphisms. Continuous morphisms are additive, i.e. $\mathcal{C}_X^{op}(X,Y)$ is a (full) subcategory of $\mathcal{C}_X^{op}(X,Y)$.

4.7.1. Proposition. Let $X$ be an object of $\mathcal{C}_X^{op}$ such that $C_X$ is an abelian category with small coproducts. Then for any unital associative ring $R$, the inclusion functor $\mathcal{C}_X^{op}(X,\text{Sp}(R)) \rightarrow \mathcal{C}_X^{op}(X,\text{Sp}(R))$ has a right adjoint.

Proof. Since $C_X$ is an abelian category with small coproducts (or, what is the same, colimits of small diagrams), it follows from 4.5.1 that for any object $O \in \text{Ob}C_X$, the functor $f_{O*} = C_X(O,-) : C_X \rightarrow \Gamma_XO - \text{mod}$

\begin{equation}
(6)
\end{equation}
is a direct image functor of a continuous morphism $f_O : X \rightarrow \text{Sp}(\Gamma_XO)$. To every morphism $X \rightarrow \text{Sp}(R)$, we assign the composition of the morphism $f_O$, where $O = f^*(R)$, and the morphism $\text{Sp}(\Gamma_XO) \rightarrow \text{Sp}(R)$ corresponding to the right $R$-module structure $\phi_f : R \rightarrow \Gamma_XO$. The assertion follows from 4.6. Details are left to the reader. ■
4.7.2. Corollary. Let $R$, $S$ be associative unital rings. The functor

\[ (S, R) \text{- bimod} \longrightarrow \text{Cat}^{op}(\text{Sp}(S), \text{Sp}(R)) \quad (7) \]

which assigns to each $(S, R)$-bimodule $\mathcal{M}$ the morphism $f_{\mathcal{M}}$ with the inverse image functor $\mathcal{M} \otimes_R - : R \text{- mod} \longrightarrow S \text{- mod}$ has a right adjoint.

Proof. The assertion follows from 4.7.1 and the fact that the functor (7) establishes an equivalence between the category $(S, R)$-bimodules and the category of continuous morphisms from $(\text{Sp}(S)$ to $\text{Sp}(R))$ (see 4.6.1).

4.8. Grothendieck categories and flat localizations of spectra of rings. Recall that a Grothendieck category is an abelian category with small coproducts, exact inductive limits, and cogenerators.

Let $C_X$ be a Grothendieck category. By 4.6, every object $O$ of $C_X$ defines a continuous morphism $f_O : X \longrightarrow \text{Sp}(\Gamma_X O)$ with a direct image functor $C_X(O, -)$. Here $\Gamma_X O$ denotes the ring $C_X(O, O)$.

4.8.1. Proposition. Let $C_X$ be a Grothendieck category. A continuous morphism $X \xrightarrow{f} \text{Sp}(R)$ is a flat localization iff the corresponding right $R$-module $(O, \phi_f)$ (cf. 4.6) has the following properties:

(a) $O$ is a cogenerator of the category $C_X$;
(b) the ring morphism $\phi_f : R \longrightarrow \Gamma_X O$ is flat (i.e. it makes $\Gamma_X O \simeq f_* f^*(R)$ a flat right $R$-module), and the morphism $\Gamma_X O \otimes_R \Gamma_X O \longrightarrow \Gamma_X O$ induced by the multiplication on $\Gamma_X O$ is an isomorphism.

Proof. 1) Suppose the conditions (a) and (b) hold.
(a) If $C_X$ is a Grothendieck category, then the morphism $f_O : X \longrightarrow \text{Sp}(\Gamma_X O)$ with the direct image functor given by $C_X(O, -)$ is a localization iff $O$ is a cogenerator of the category $C_X$.

This assertion follows from the argument of the Theorem by Gabriel and Popescu which describes Grothendieck categories as quotient categories (localizations) of categories of modules. See [BD, 6.25].

(b) The condition (b) means exactly that the morphism $\text{Sp}\Gamma_X O \longrightarrow \text{Sp}(R)$ corresponding to the ring morphism $\phi_f$ is a localization.

Indeed, the (right) flatness of $\phi_f : R \longrightarrow \Gamma_X O$ means that the corresponding inverse image functor, $\phi_f^* = \Gamma_X O \otimes_R -$ , is exact. The condition that $\Gamma_X O \otimes_R \Gamma_X O \longrightarrow \Gamma_X O$ is an isomorphism implies that the adjunction morphism $\phi_f^* \phi_f^* \longrightarrow Id_{\Gamma_X O - \text{mod}}$ is an isomorphism. The latter is equivalent to that $\phi_f^*$ is fully faithful, i.e. $\phi_f^*$ is a localization.

Thus, $f$, being a composition of flat localizations is a flat localization.

2) Suppose $X \xrightarrow{f} \text{Sp}(R)$ is a localization.
(i) For any continuous morphism $X \xrightarrow{f} \text{Sp}(R)$, the object $O = f^*(R)$ is a cogenerator iff $f_*$ is a faithful functor.

In fact, by definition, the object $O$ is a cogenerator iff the functor $C_X(O, -)$ is faithful. But, this functor is isomorphic to $f_*$. 26
A continuous morphism $f$ is a localization iff its direct image functor, $f_*$, is fully faithful; in particular, $\mathcal{O}$ is a cogenerator.

(ii) Since $\mathcal{O}$ is a cogenerator, the morphism $f_\mathcal{O} : X \to \text{Sp}(\Gamma X \mathcal{O})$ is a flat localization. Let $\bar{\phi}_f$ denote the morphism $\text{Sp}(\Gamma X \mathcal{O}) \to \text{Sp}(R)$ corresponding to the ring morphism $\phi_f$.

Since the composition $f = \bar{\phi}_f \circ f_\mathcal{O}$ is a localization and $f_\mathcal{O}$ is a localization, it follows from the universal property of localizations that $\bar{\phi}_f$ is a localization. The latter means that the adjunction morphism $\bar{\phi}^* \bar{\phi}_s(M) = \Gamma X \mathcal{O} \otimes_R \phi_*(M) \to M$ is an isomorphism for any $\Gamma X \mathcal{O}$-module $M$. Taking $M = \Gamma X \mathcal{O}$, we obtain that the natural morphism $\Gamma X \mathcal{O} \otimes_R \Gamma X \mathcal{O} \to \Gamma X \mathcal{O}$ is an isomorphism.

(iii) It remains to verify that $\Gamma X \mathcal{O}$ is flat as a right $R$-module. This is a consequence of the following general fact:

4.8.1.1. Lemma. Let $X \xrightarrow{g} Y \xrightarrow{h} Z$ be continuous localizations. Suppose that the category $C_Z$ has finite limits and that inverse image functors of $g$ and $h \circ g$ are left exact. Then $h^*$ is left exact.

Let $D : \mathcal{O} \to C_Y$ be a finite diagram. Since $h^*$ is a localization, the functor $h_*$ is fully faithful. By [GZ, I.1.4], $\lim D$ exists if $\lim h_* D$ exists, and the natural morphism $h^*(\lim h_* D) \to \lim h^* h_* D \simeq \lim D$ is an isomorphism. Therefore

$$g^*(\lim D) \simeq g^* h^* (\lim h_* D) \simeq (hg)^* (\lim h_* D).$$

Since the functor $(hg)^*$ is exact,

$$(hg)^* (\lim h_* D) \simeq \lim ((hg)^* h_* D) \simeq \lim (g^* h^* h_* D) \simeq \lim (g^* D)$$

(the last isomorphism is induced by the adjunction isomorphism $h^* h_* \to Id_{C_Y}$). Hence the assertion. ■ ■

4.8.1.2. Note. The requirement in 4.8.1.1 that localizations are continuous can be dropped. The argument in this case is a little bit more involved. It is based on the fact that if $X \xrightarrow{f} Y$ is a localization and the category $C_Y$ has finite limits, then the functor $f^*$ is exact iff the class of morphisms $\Sigma_f = \{ s \in HomC_Y \mid f^*(s) \text{ is an isomorphism} \}$ satisfies the right Ore conditions (see 3.1.1, or [GZ, I.3.1 and 1.3.4]).

4.8.2. Corollary. Let $C_X$ be a Grothendieck category. Any continuous morphism $X \xrightarrow{f} \text{Sp}(R)$ such that $f_*$ is a faithful functor is uniquely represented as the composition

$$X \xrightarrow{\psi_f} \text{Sp}(R') \xrightarrow{\bar{\phi}_f} \text{Sp}(R),$$

(8)

where $\bar{\phi}_f$ is an affine morphism corresponding to a ring morphism $\phi_f : R \to R'$, and $\psi_f$ is a flat localization such that the adjunction morphism $R' \to \psi_{f*} \psi_f^*(R')$ is an isomorphism.

The morphism $f$ is a localization (resp. a flat localization) iff $\bar{\phi}_f$ is a localization (resp. a flat localization).

Proof. Let $\mathcal{O} = f^*(R)$. The functor $f_*$ is faithful iff the object $\mathcal{O}$ is a cogenerator in $C_X$. By (the argument of) Gabriel-Popescu Theorem, this implies that the canonical
morphism \( f_0 : X \rightarrow \text{Sp}(\Gamma_X \mathcal{O}) \) is a flat localization. We set \( \psi_f = f_0 \) and \( R' = \Gamma_X \mathcal{O} = C_X(\mathcal{O}, \mathcal{O})^\circ \). The morphism \( \phi_f : R \rightarrow R' \) is the right \( R \)-module structure on \( \mathcal{O} \) (which is the same as the adjunction morphism \( R \rightarrow f_*f^*(R) \) if the ring \( C_X(\mathcal{O}, \mathcal{O})^\circ \) is identified with \( f_*f^*(R) \)). It follows from 4.8.1 that if \( f \) is a localization (i.e. \( f_* \) is fully faithful), then \( \phi_f \) is a localization.

The condition \( R' \simeq \psi_f \psi^*_f(R') \) implies that the ring \( R' \) is isomorphic to \( \Gamma_X \mathcal{O} \), where \( \mathcal{O} = f^*(R) \). This implies uniqueness (up to isomorphism) of the decomposition (8). \( \blacksquare \)

5. Monads, comonads, and continuous morphisms.

5.1. Monads and their categoric spectrum. Let \( Y \) be an object of \( |\text{Cat}|^\circ \). A monad on \( Y \) is by definition a monad on the category \( C_Y \), i.e. a pair \((F, \mu)\), where \( F \) is a functor \( C_Y \rightarrow C_Y \) and \( \mu \) a morphism \( F^2 \rightarrow F \) (multiplication) such that \( \mu \circ F \mu = \mu \circ \mu \) and there exists a morphism \( \eta : \text{id}_{C_Y} \rightarrow F \) uniquely determined by the equalities \( \mu \circ F \eta = \text{id}_F = \mu \circ \eta F \) (a unit).

A morphism from a monad \( \mathcal{F} = (F, \mu) \) to a monad \( \mathcal{F}' = (F', \mu') \) is given by a functor morphism \( F \rightarrow F' \) such that \( \phi \circ \mu = \mu' \circ \phi \circ \phi \) and \( \phi \circ \eta = \eta' \). Here \( \phi \circ \phi = F' \phi \circ \phi F \), and \( \eta, \eta' \) are units of the monads resp. \( \mathcal{F} \) and \( \mathcal{F}' \). The composition of morphisms is defined naturally, so that the map \( \text{Mon}_Y \rightarrow \text{End}(C_Y) \) forgetting monad structure, i.e. sending a monad morphism \( (F, \mu) \xrightarrow{\phi} (F', \mu') \) to the natural transformation \( F \xrightarrow{\phi} F' \), is a functor.

For an object \( Y \) of \( |\text{Cat}|^\circ \), we denote by \( \text{Mon}_Y \) the category of monads on \( Y \).

Given a monad \( \mathcal{F} = (F, \mu) \) on \( Y \), we denote by \((\mathcal{F}/Y) - \text{mod}\), or simply by \( \mathcal{F} - \text{mod}\), the category of \((\mathcal{F}/Y)\)-modules. Its objects are pairs \( (M, \xi) \), where \( M \in \text{Ob}C_Y \) and \( \xi \) a morphism \( F(M) \rightarrow M \) such that \( \xi \circ F \xi = \xi \circ \mu(M) \) and \( \xi \circ \eta(M) = \text{id}_M \). Morphisms from \((M, \xi)\) to \((M', \xi')\) are given by morphisms \( g : M \rightarrow M' \) such that \( \xi' \circ Fg = g \circ \xi \).

We denote by \( \text{Sp}(\mathcal{F}/Y) \) the object of \( |\text{Cat}|^\circ \) such that the corresponding category is the category \((\mathcal{F}/Y) - \text{mod}\) of \((\mathcal{F}/Y)\)-modules. The object \( \text{Sp}(\mathcal{F}/Y) \) is regarded as the spectrum of the monad \( \mathcal{F} \) in \( |\text{Cat}|^\circ \).

The forgetful functor

\[
(\mathcal{F}/Y) - \text{mod} \xrightarrow{f_*} C_Y, \quad (M, \xi) \mapsto M,
\]

is a right adjoint to the functor

\[
C_Y \xrightarrow{f^*} (\mathcal{F}/Y) - \text{mod}, \quad L \mapsto (F(L), \mu(L)), \quad (L \xrightarrow{g} N) \mapsto (f^*(L) \xrightarrow{F(g)} f^*(L')).
\]

In other words, we have a canonical continuous morphism \( \text{Sp}(\mathcal{F}/Y) \xrightarrow{f} Y \).

5.1.1. Example. Let \( R, S \) be unital associative rings. Any unital ring morphism \( \varphi : S \rightarrow R \) defines a monad, \( R^\varphi = (R^\varphi, \mu^\varphi) \), on \( Y = \text{Sp}(S) \). Here the functor \( R^\varphi \) is \( M \mapsto R \otimes_S M \), and the multiplication is induced by the multiplication on \( R \). The canonical morphism \( \text{Sp}(R/\text{Sp}(S)) \rightarrow \text{Sp}(S) \) has the pull-back functor \( \phi_* : R - \text{mod} \rightarrow S - \text{mod} \) as a direct image functor. Notice that the category \((R^\varphi/\text{Sp}(S))\)-modules is isomorphic to the category \( R - \text{mod} \) of \( R \)-modules; in particular, \( \text{Sp}(R^\varphi/\text{Sp}(S)) \simeq \text{Sp}(R) \).

28
If \( S = \mathbb{Z} \), i.e. \( C_Y = \mathbb{Z} - mod \), the category \((R/\text{Sp}\mathbb{Z}) - mod\) coincides with the category \( R - mod \) of left \( R \)-modules. In this case, consistently with our previous notations, we write \( \text{Sp}(R) \) instead of \( \text{Sp}(R/\text{Sp}\mathbb{Z}) \).

5.1.2. Example. Any monoid morphism \( M \xrightarrow{\phi} cN \) defines a monad, \( \mathcal{F} = (F, \mu) \), on \( Y = \text{Sp}(M/E) \), where the functor \( F \) is \( M \boxtimes N \). A \( \text{left } M \)-set \((L, \xi)\) to the \( cokernel \) of the pair of morphisms \( N \times M \times L \xrightarrow{\phi \times L} N \times \xi \) and another is the composition of \( N \times M \times L \xrightarrow{\nu \times L} N \times L \). Here \( N \times N \xrightarrow{\nu} N \) is the multiplication on \( N \). The multiplication \( F \mu \) is induced by the multiplication on \( N \). The canonical morphism \( \text{Sp}(\mathcal{F}/\text{Sp}(M/E)) \xrightarrow{\phi \ast} \mathcal{M} - sets \) as a direct image functor.

5.1.3. Example: localizations of modules. Let \( R \) be an associative unital ring and \( \mathfrak{F} \) a set of left ideals in \( R \). Denote by \( R - mod \mathfrak{F} \) the full subcategory of \( R - mod \) whose \( \text{objects are } \) \( R \)-modules \( M \) such that the canonical morphism \( M \xrightarrow{\phi} \text{Hom}_R(m, M) \) is an isomorphism for all \( m \in \mathfrak{F} \). The inclusion functor \( \mathcal{R} - mod \xrightarrow{j_\mathfrak{F} \ast} \mathcal{R} - mod \) preserves limits, hence it has a left adjoint, \( j_\mathfrak{F} \). Since \( j_\mathfrak{F} \) is fully faithful, \( j_\mathfrak{F}^* \) is a localization. The \( R \)-module \( R_\mathfrak{F} = j_\mathfrak{F}^* j_\mathfrak{F}^*(R) \) has a structure of a ring uniquely determined by the fact that the adjunction arrow, \( R \xrightarrow{\eta_\mathfrak{F}} R_\mathfrak{F} \) is a ring morphism. There is a canonical functor morphism \( \tau_\mathfrak{F} \).

Suppose all ideals in \( \mathfrak{F} \) are projective modules. Then the inclusion functor \( \mathcal{R} - mod \xrightarrow{j_\mathfrak{F} \ast} \mathcal{R} - mod \) is exact. This implies that the morphism

\[
R_\mathfrak{F} \otimes_R M \xrightarrow{\tau_\mathfrak{F}^*(M)} j_\mathfrak{F}^* j_\mathfrak{F}^*(M)
\]

is an isomorphism for any module \( M \) of finite type.

If \( \mathfrak{F} \) consists of projective ideals of finite type, then \( \mathcal{R} - mod \xrightarrow{j_\mathfrak{F} \ast} \mathcal{R} - mod \) is an affine morphism, or, equivalently, the functor morphism \( \tau_\mathfrak{F} \) is an isomorphism. Thus the category \( \mathcal{R} - mod \mathfrak{F} \) is equivalent to the category \( \mathcal{R} - mod \mathfrak{F} \) of left \( R_\mathfrak{F} \)-modules.

5.1.3.1. Note. In general, the localization \( j_\mathfrak{F} \) is not flat, i.e. the functor \( j_\mathfrak{F}^* \) is not exact. Denote by \( \mathfrak{F}^- \) the set of all left ideals of the ring \( R \) such that the canonical morphism \( M \xrightarrow{\phi} \text{Hom}_R(m, M) \) is an isomorphism for all \( M \in \text{Ob}R - mod \mathfrak{F} \). Clearly \( R - mod \mathfrak{F}^- = R - mod \mathfrak{F} \). It follows from results of Gabriel (cf. [Gab], or [BD, Ch. 6]) that the localization \( j_\mathfrak{F} \) is flat iff \( \mathfrak{F}^- \) is a radical filter; i.e. with any left ideal \( m \), the set
\( \mathfrak{F}^- \) contains left ideals \( (m : r) = \{ a \in R \mid ar \in m \} \) for all \( r \in R \) and all left ideals \( n \) in \( R \) such that \( (n : r) \in \mathfrak{F}^- \) for all \( r \in m \). These conditions are equivalent to that the full subcategory \( T_{\mathfrak{F}^-} \) of \( R - \text{mod} \) whose objects are all \( R \)-modules \( M \) such that every element of \( M \) is annihilated by some ideal \( m \in \mathfrak{F}^- \), is a Serre subcategory.

5.1.4. Curves. Let \( R \) be a ring of the homological dimension one, or, equivalently, every left ideal in \( R \) is projective. Then for any set of left ideals \( \mathfrak{F} \), the inclusion functor (2) is exact. If, in addition, \( R \) is left noetherian, the functor (2) is strictly exact.

5.2. Morphisms of monads and morphisms of their categoric spectra. Let \( Y \) be an object of \( |\text{Cat}|^o \) and \( \mathcal{F}, \mathcal{F}' \) monads on \( Y \). Any monad morphism \( \mathcal{F} \xrightarrow{\varphi} \mathcal{F}' \) induces the 'pull-back' functor

\[
(\mathcal{F}' / Y) - \text{mod} \xrightarrow{\varphi^*} (\mathcal{F} / Y) - \text{mod}, \quad (M, \xi) \mapsto (M, \xi \circ \varphi(M)).
\]

This correspondence defines a functor \( \text{Mon}^w_Y : \mathcal{F} \xrightarrow{\varphi} \mathcal{F}' \) that takes values in the full subcategory of \( \mathcal{F} \mathcal{C} / \mathcal{C}_Y \) objects of which are functors \( \mathcal{C}_Z \rightarrow \mathcal{C}_Y \) having a left adjoint.

5.2.1. Reflexive pairs of arrows and weakly continuous functors and monads. Recall that a pair of arrows \( M \xrightarrow{g_1} L \) in \( \mathcal{C}_Y \) is called reflexive, if there exists a morphism \( L \xrightarrow{h} M \) such that \( g_1 \circ h = \text{id}_M = g_2 \circ h \).

We call a functor \( \mathcal{C}_Y \rightarrow \mathcal{C}_Z \) weakly continuous if it preserves cokernels of reflexive pairs of arrows.

We call a monad \( \mathcal{F} = (\mathcal{F}, \mu) \) on \( Y \) weakly continuous if the functor \( \mathcal{C}_Y \xrightarrow{\mathcal{F}} \mathcal{C}_Y \) is weakly continuous. We denote by \( \text{Mon}^w_Y \) the full subcategory of the category \( \text{Mon}_Y \) whose objects are weakly continuous monads on \( Y \).

5.2.2. Lemma. Let \( \mathcal{F}, \mathcal{F}' \) be monads on \( Y \) and \( \varphi \) a monad morphism \( \mathcal{F} \rightarrow \mathcal{F}' \). Suppose the category \( \mathcal{C}_Y \) has cokernels of reflexive pairs of morphisms and the monad \( \mathcal{F}' \) is weakly continuous. Then the functor \( \varphi_* \) has a left adjoint.

In particular, the map \( (\mathcal{F} / Y) \leftrightarrow \text{Sp}(\mathcal{F} / Y), \varphi \mapsto [\varphi^*] \) is a functor,

\[
\text{Sp}_Y : \text{Mon}^w_Y \longrightarrow |\text{Cat}|^o,
\]

which takes values in the subcategory \( |\text{Cat}|^o_{\text{cont}} \) of \( |\text{Cat}|^o \) formed by continuous morphisms.

Proof. The left adjoint, \( (\mathcal{F} / Y) - \text{mod} \xrightarrow{\varphi^*} (\mathcal{F}' / Y) - \text{mod} \) assigns to each \( (\mathcal{F} / Y) \)-module \( (M, \xi \circ F(M) \xrightarrow{\xi} M) \) the cokernel of the pair of arrows

\[
F'F(M) \xrightarrow{\mu' \circ F' \varphi} F(M).
\]

Since by hypothesis \( F' \) preserves cokernels of reflexive pairs and both arrows (1) are \( \mathcal{F}' \)-module morphisms, there exists a unique \( \mathcal{F}' \)-module structure on the cokernel of (1). Details are left to the reader. \( \blacksquare \)
5.2.3. Note. Suppose that the category $C_X$ has colimits of certain type $\mathcal{D}$, and let $\mathcal{F} = (F, \mu)$ be a monad on $X$ such that the functor $F$ preserves colimits of this type. Then the category $(\mathcal{F}/X) - \text{mod}$ has colimits of this type.

In fact, for a diagram $\mathcal{D} \xrightarrow{\delta} (\mathcal{F}/X) - \text{mod}$, the colimit of the composition $f_\delta \circ \delta$ (where $f_\delta$ is the forgetful functor $(\mathcal{F}/X) - \text{mod} \rightarrow C_X$) has a unique $\mathcal{F}$-module structure, $\xi_\mathcal{D}$. The $\mathcal{F}$-module $(\text{colim}(f_\delta \circ \delta), \xi_\mathcal{D})$ is a colimit of the diagram $\mathcal{D}$.

In particular, if $\mathcal{F} = (F, \mu)$ is a weakly continuous monad on $X$, and the category $C_X$ has cokernels of reflexive pairs of arrows, then the category $(\mathcal{F}/X) - \text{mod}$ has cokernels of reflexive pairs of arrows.

5.3. Comonads and their cospectrum. A comonad on $Y$ is a monad on the dual object (‘space’) $Y^\circ$ defined by $C_{Y^\circ} = C_Y^{op}$. In other words, a comonad on $Y$ is a pair $(G, \delta)$, where $G$ is a functor $C_Y \rightarrow C_Y$ and $\delta$ a functor morphism $G \rightarrow G^2$ (a comultiplication) such that $G\delta \circ \delta = \delta G \circ \delta$ and $G\epsilon \circ \delta = \epsilon G \circ \delta$ for a uniquely determined morphism $G \xrightarrow{\epsilon} \text{Id}_{C_Y}$ (a counit).

We denote the category of comonads on $Y$ by $\mathcal{CMon}_Y$. It is defined by the formula $\mathcal{CMon}_Y = \mathcal{Mon}_Y^{op}$.

Comodules over a comonad $\mathcal{G} = (G, \delta)$ are just modules over the dual monad on $Y^\circ$. In terms of $Y$, a $\mathcal{G}$-comodule is a pair $(M, \xi)$, where $M \in ObC_Y$ and $\xi$ a morphism $M \rightarrow G(M)$ such that $\delta(M) \circ \xi = G\xi \circ \xi$ and $\epsilon(M) \circ \xi = \text{Id}_M$. We denote the category of comodules over $\mathcal{G}$ by $(Y\backslash \mathcal{G}) - \text{Comod}$, or simply by $\mathcal{G} - \text{Comod}$.

We denote by $\mathcal{Sp}^o(Y\backslash \mathcal{G})$ the object of $|\text{Cat}|^o$ (or $\text{Cat}^{op}$) such that the corresponding category is $(Y\backslash \mathcal{G}) - \text{Comod}$. This definition can be rephrased as follows:

$$\mathcal{Sp}^o(Y\backslash \mathcal{G}) = \text{Sp}(G^o/Y^\circ)^o. \quad (1)$$

Here $G^o$ is the monad $(G^o, \delta^o)$ on $Y^\circ$ dual to the comonad $\mathcal{G}$.

We call $\mathcal{Sp}^o(Y\backslash \mathcal{G})$ the cospectrum of the comonad $\mathcal{G}$ in $|\text{Cat}|^o$.

By duality, there is a canonical continuous morphism $Y \xrightarrow{g^*} \mathcal{Sp}^o(Y\backslash \mathcal{G})$ with an inverse image functor

$$(Y\backslash \mathcal{G}) - \text{Comod} \xrightarrow{g_*} C_Y, \quad (M, \xi) \mapsto M, \quad (2)$$

and having a direct image functor

$$C_Y \xrightarrow{g_*} (Y\backslash \mathcal{G}) - \text{Comod}, \quad L \mapsto (G(L), \delta(L)). \quad (3)$$

5.3.1. Example. Let $R$ be an associative unital ring and $\mathcal{H} = (H, \delta)$ a coalgebra in the monoidal category of $R$-bimodules. This means that $H$ is an $R$-bimodule, $\delta$ an $R$-bimodule morphism $H \rightarrow H \otimes_R H$ such that $\delta \otimes_R \text{Id}_H \circ \delta = \text{Id}_H \otimes_R \delta \circ \delta$, and there exists a (necessarily unique) $R$-bimodule morphism $\epsilon : H \otimes_R H \rightarrow R$ such that $\lambda_r(H) \circ \epsilon \otimes_R \text{Id}_H \circ \delta = \text{Id}_H = \lambda_l(H) \circ \text{Id}_H \otimes_R \epsilon \circ \delta$. Here $\lambda_l(H) : R \otimes_R H \rightarrow H$ and $\lambda_r(H) : H \otimes_R H \rightarrow H$ are canonical isomorphisms. The coalgebra $\mathcal{H}$ induces a comonad on the category $R - \text{mod}$ of left $R$-modules tensoring by $H$ over $R$, $L \mapsto H \otimes_R L$, as a functor and with comultiplication $H \otimes_R - \rightarrow H \otimes_R H \otimes_R -$ induced by the comultiplication $\delta$. 

31
The canonical morphism \( \text{Sp}(R) \to \text{Sp}^\circ(\text{Sp}(R) \backslash \mathcal{H}) \) has the forgetful functor \( (\text{Sp}(R) \backslash \mathcal{H}) \to \text{Comod} \to R \text{mod} \) as an inverse image functor.

5.3.2. Functoriality of the cospectrum. Let \( \mathcal{G} = (G, \delta) \) and \( \mathcal{G}' = (G', \delta') \) be comonads on \( Y \) and \( \psi \) a comonad morphism \( \mathcal{G} \to \mathcal{G}' \); i.e. \( \psi \) is a morphism of functors \( G \to G' \) such that \( \delta' \circ \psi = \psi \circ \psi \circ \delta \) and \( \epsilon' \circ \psi = \epsilon \). Here \( \psi \circ \psi = G' \circ \psi \circ G \), and \( \epsilon, \epsilon' \) are counits of the comonads resp. \( \mathcal{G} \) and \( \mathcal{G}' \). The morphism \( \psi \) induces the ‘pull-back’ functor

\[
\begin{array}{c}
(Y \backslash \mathcal{G}) \to \text{Comod} \\
\psi^* \\
\end{array}
\]

which is regarded as an inverse image functor of a morphism

\[
\text{Sp}^\circ(\psi) : \text{Sp}^\circ(Y \backslash \mathcal{G}') \to \text{Sp}^\circ(Y \backslash \mathcal{G})
\]

The map (4) defines a functor

\[
\tilde{\text{Sp}}^\circ_Y : \text{CMon}_Y^{\text{op}} \longrightarrow (C_Y \backslash \text{Cat}^{\text{op}})_\epsilon,
\]

where \( (C_Y \backslash \text{Cat}^{\text{op}})_\epsilon \) denotes the full subcategory of the category \( C_Y \backslash \text{Cat}^{\text{op}} \) whose objects are continuous morphisms.

5.3.2.1. Proposition. The functor (6) is fully faithful and has a right adjoint.

Proof. Let

\[
\begin{array}{c}
X \\
h \\
\downarrow f \\
Y \nearrow g \\
Z
\end{array}
\]

be a morphism in \( (C_Y \backslash \text{Cat}^{\text{op}})_\epsilon \) given by the commutative diagram

\[
\begin{array}{c}
C_Z \\
h^* \\
\downarrow f^* \\
C_Y \\
g^*
\end{array} \quad \begin{array}{c}
C_X \\
g^* \\
\downarrow f^*
\end{array}
\]

of functors. Fix direct image functors of \( f \) and \( g \) and the corresponding adjunction arrows. Set

\[
\varphi_h = f^* (f_* \epsilon_g \circ \eta_f h^* g_*) : g^* g_* \longrightarrow f^* f_*.
\]

One can check that \( \varphi_h \) is a monad morphism \( \mathcal{G}_g \to \mathcal{G}_f \) and the map \( \tilde{\Gamma}_Y : h \mapsto \varphi_h \) is functorial. The composition \( \tilde{\Gamma}_Y \circ \tilde{\text{Sp}}^\circ_Y \) is the identical functor which provides one of the adjunction arrows and shows that the functor \( \tilde{\text{Sp}}^\circ_Y \) is fully faithful. We leave the other adjunction arrow to the reader (it is defined in 5.4). ■

Recall that pair of arrows \( M \xrightarrow{g_1} L \) is called coreflexive, if there exists a morphism \( L \xrightarrow{h} M \) such that \( h \circ g_1 = \text{id}_M = h \circ g_2 \).
5.3.2.2. Lemma. Suppose that the category \( C_Y \) has kernels of coreflexive pairs of morphisms and the functor \( G \) preserves these kernels. Then the functor \( \psi^* \) has a right adjoint, i.e. the morphism (4) is continuous.

Proof. The assertion is the dual version of 5.2.2. ■

5.4. Beck’s theorem. Let \( X \xrightarrow{f} Y \) be a continuous morphism in \(|\text{Cat}|^o\) with inverse image functor \( f^* \), direct image functor \( f_* \), and adjunction morphisms

\[
\begin{align*}
\text{Id}_{C_Y} & \xrightarrow{\eta_f} f_*f^* \quad \text{and} \quad f^*f_* \xrightarrow{\epsilon f} \text{Id}_{C_X}.
\end{align*}
\]

Let \( G_f \) denote the comonad \((G_f, \delta_f)\), where \( G_f = f^*f_* \) and \( \delta_f = f^*\eta_f f_* \). There is a commutative diagram

\[
\begin{array}{ccc}
C_Y & \xrightarrow{\tilde{f}^*} & (X \setminus G_f) - \text{Comod} \\
\downarrow{f^*} & & \swarrow{\tilde{f}^*} \\
C_X & & \\
\end{array}
\]

Here \( \tilde{f}^* \) is the canonical functor \( C_Y \rightarrow (X \setminus G_f) - \text{Comod} \), \( M \mapsto (f^*(M), f^*\eta_f(M)) \), and \( \tilde{f}^* \) is the forgetful functor \((X \setminus G_f) - \text{Comod} \rightarrow C_X \). The diagram \((1^o)\) is regarded as the diagram of inverse image functors of the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\tilde{f}} & \mathbf{Sp}^o(X \setminus G_f) \\
\downarrow{f} & & \swarrow{\tilde{f}} \\
Y & & \\
\end{array}
\]

in \(|\text{Cat}|^o\). The following statement is one of the versions of the Beck’s theorem.

5.4.1. Theorem. Let \( X \xrightarrow{f} Y \) be a continuous morphism.

(a) If the category \( C_X \) has kernels of coreflexive pairs of arrows, then the functor \( \tilde{f}^* \) has a right adjoint, \( \tilde{f}_* \), i.e. \( \mathbf{Sp}^o(X \setminus G_f) \xrightarrow{\tilde{f}} Y \) is a continuous morphism.

(b) If, in addition, \( f \) is weakly flat, i.e. the functor \( f^* \) preserves kernels of coreflexive pairs, then the adjunction arrow \( \tilde{f}^* \tilde{f}_* \xrightarrow{\epsilon \tilde{f}} \text{Id}_{(X \setminus G_f) - \text{Comod}} \) is an isomorphism, i.e. \( \tilde{f}_* \) is a fully faithful functor, or, equivalently, \( \tilde{f}^* \) is a localization.

(c) If, in addition to (a) and (b), \( f^* \) reflects isomorphisms, then the adjunction arrow \( \text{Id}_{C_Y} \xrightarrow{\eta_f} f_*f^* \) is an isomorphism too, i.e. \( \tilde{f} \) is an isomorphism.

Proof. See [MLM], IV.4.2, or [ML], VI.7. ■

We need also the dual version of the theorem 5.4.1. Let \( F_f \) denote the monad \((F_f, \mu_f)\), where \( F_f = f_*f^* \) and \( \mu_f = f_*\epsilon_f f^* \). There is a commutative diagram

\[
\begin{array}{ccc}
C_X & \xrightarrow{\tilde{f}_*} & (F_f/Y) - \text{mod} \\
\downarrow{f_*} & & \swarrow{\tilde{f}_*} \\
C_Y & & \\
\end{array}
\]
Here $\tilde{f}_*$ is the canonical functor

$$C_X \rightarrow (\mathcal{F}_f/Y) - \text{mod}, \quad M \mapsto (f_*(M), f_!\epsilon_f(M)),$$

$\tilde{f}_*$ the forgetful functor $(\mathcal{F}_f/Y) - \text{mod} \rightarrow C_Y$.

5.4.2. Theorem. Let $X \xrightarrow{f} Y$ be a continuous morphism.

(a) If the category $C_Y$ has cokernels of reflexive pairs of arrows, then the functor $\tilde{f}_*$ has a left adjoint, $\tilde{f}^*$; hence $\tilde{f}_*$ is a direct image functor of a continuous morphism $\tilde{X} \xrightarrow{\tilde{f}} \text{Sp}(\mathcal{F}_f/Y)$.

(b) If, in addition, the functor $f_*$ preserves cokernels of reflexive pairs, then the adjunction arrow $\tilde{f}^*\tilde{f}_* \xrightarrow{\sim} \text{Id}_{C_X}$ is an isomorphism, i.e. $\tilde{f}_*$ is a localization.

(c) If, in addition to (a) and (b), the functor $f_*$ is conservative, then $\tilde{f}_*$ is a category equivalence.

Proof. The theorem is dual (hence equivalent) to the theorem 5.4.1. ■

If the condition (a) in 5.4.2 holds, then to the diagram (1), there corresponds a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\tilde{f}} & \text{Sp}(\mathcal{F}_f/Y) \\
\downarrow f & \nearrow \tilde{f} & \downarrow j \\
Y & & \\
\end{array}
$$

in $|\text{Cat}|^\circ$. If the condition (c) in 5.4 holds, the morphism $\tilde{f}$ in (2) is an isomorphism.

Thus, given a continuous morphism $X \xrightarrow{f} Y$ such that the category $C_Y$ has cokernels of reflexive pairs of arrows, we have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\tilde{f}} & \text{Sp}(\mathcal{F}_f/Y) \\
\downarrow f & \nearrow \tilde{f} & \downarrow j \\
\text{Sp}^\circ(X \setminus \mathcal{G}_f) & \xrightarrow{\tilde{f}} & Y \\
\end{array}
$$

Notice that the diagrams (1) and (1*) are uniquely defined by the data $(f^*, f_*, \epsilon_f, \eta_f)$, since the monad $\mathcal{F}_f$, the comonad $\mathcal{G}_f$, and the functors $\tilde{f}_*: C_X \rightarrow (\mathcal{F}_f/Y) - \text{mod}$ in (1) and $\tilde{f}^*: C_Y \rightarrow (X \setminus \mathcal{G}_f) - \text{Comod}$ are defined in terms of this data. Given the functor $f^*$ (resp. $f_*$), the rest of the data, $f_*, \epsilon_f, \eta_f$ (resp. $f^*, \epsilon_f, \eta_f$), is determined uniquely up to isomorphism. Thus, the monad $\mathcal{F}_f$ and the comonad $\mathcal{G}_f$ in the diagrams (1) and (1*) are determined by $f^*$ uniquely up to isomorphism.

5.5. Weakly flat and weakly affine morphisms. We call a functor weakly continuous (resp. weakly flat) if it preserves cokernels of reflexive pairs of arrows (resp. kernels of coreflexive pairs of arrows).

We call a monad $(F, \mu)$ weakly continuous if the functor $F$ is weakly continuous, and a comonad $(G, \delta)$ weakly flat if the functor $G$ is weakly flat.

Let $\text{Mon}_X^w$ denote the full subcategory of the category $\text{Mon}_X$ of monads on $X$ spanned by weakly continuous monads, and let $\text{CMon}_X^w$ be the full subcategory of the category $\text{CMon}_X$ of comonads on $X$ spanned by weakly flat comonads on $X$. 

34
We call a continuous morphism $X \xrightarrow{f} Y$ weakly affine if its direct image functor is weakly continuous and the category $C_X$ has cokernels of reflexive pairs of arrows. Let $\text{Aff}^w_Y$ denote the full subcategory of the category $|\text{Cat}|^o/Y$ whose objects are weakly affine morphisms to $Y$.

We call a continuous morphism $X \xrightarrow{f} Y$ in weakly flat if its inverse image functor is weakly flat and the category $C_Y$ has kernels of coreflexive pairs of arrows. We denote by $\text{Flat}^w_X$ the full subcategory of the category $X\setminus|\text{Cat}|^o$ whose objects are weakly flat morphisms from $X$.

5.5.1. Proposition. (a) Suppose the category $C_Y$ has cokernels of reflexive pairs of arrows. The map $(\mathcal{F}/Y) \mapsto (\mathsf{Sp}(\mathcal{F}/Y) \rightarrow Y)$ defines a full functor

$$\mathsf{Sp}_Y^w : (\mathcal{Mon}_Y^w)^{\text{op}} \longrightarrow \text{Aff}^w_Y.$$ 

(b) Dually, if the category $C_Y$ has kernels of coreflexive pairs of arrows, then the map $(X\setminus\mathcal{G}) \mapsto (X \rightarrow \mathsf{Sp}^o(X\setminus\mathcal{G}))$ defines a full functor

$$\mathsf{Sp}^o_X^w : (\mathcal{CMon}_X^w)^{\text{op}} \longrightarrow \text{Flat}^w_X.$$ 

Proof. These facts follow from Beck's Theorem and the following

5.5.2. Lemma. Let $X \xrightarrow{f} Y$ be a continuous morphism with a direct image functor $f_*$ and an inverse image functor $f^*$. 

(a) Suppose the morphism $f$ is monadic and the category $C_Y$ has colimits of a type $\mathcal{S}$. Then $f_*$ preserves colimits of the type $\mathcal{S}$ iff the functor $F_f = f_* f^*$ has this property. 

(b) Dually, if the morphism $f$ is comonadic and the category $C_X$ has limits of a certain type, then $f^*$ preserves these limits iff the functor $G_f = f^* f_*$ does the same.

Recall that a continuous morphism $X \xrightarrow{f} Y$ is called comonadic if the induced morphism $\tilde{f} : \mathsf{Sp}^o(X\setminus\mathcal{G}) \rightarrow Y$ is an isomorphism.

Dually, a continuous morphisms $X \xrightarrow{f} Y$ (in $|\text{Cat}|^o$, or in $\text{Cat}^{op}$) is called monadic if the associated morphism $\tilde{X} \xrightarrow{\tilde{f}} \mathsf{Sp}(\mathcal{F}/Y)$ is an isomorphism.

The proof of the lemma and details of the proof of 5.5.1 are left to the reader. ■

5.5.3. Proposition. Let

$$\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow f & & \nearrow g \\
\downarrow Z & & \\
\end{array}$$

be a commutative diagram in $|\text{Cat}|^o$. Suppose $C_Z$ has colimits of reflexive pairs of arrows. If $f$ and $g$ are weakly affine, then $h$ is weakly affine.

Dually, if

$$\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow f & & \nearrow g \\
\downarrow Z & & \\
\end{array}$$

be a commutative diagram in $|\text{Cat}|^{\text{op}}$. Suppose $C_Z$ has kernels of coreflexive pairs of arrows. If $f$ and $g$ are weakly flat, then $h$ is weakly flat.
is a commutative diagram in $\textbf{Cat}^\circ$ such that $C_Z$ has kernels of coreflexive pairs of arrows and the morphisms $f$ and $g$ are weakly flat, then $h$ is weakly flat.

Proof. Fix inverse and direct image functors of $f$ and $g$ together with adjunction morphisms. By hypothesis, the canonical morphisms $C_X \rightarrow (\mathcal{F}_f/Z) - \text{mod}$ and $C_Y \rightarrow (\mathcal{F}_g/Z) - \text{mod}$ are category equivalences. Here $\mathcal{F}_f = (f_\ast f^\ast, \mu_f)$ and $\mathcal{F}_g = (g_\ast g^\ast, \mu_g)$ are monads associated with resp. $f$ and $g$. It follows from the dual version of 5.3.2.1 (or the argument of 10.7.1 in [R4]) that a choice of an inverse image functor $h^\ast$ of the morphism $h$ determines a monad morphism $\mathcal{F}_g \xrightarrow{\phi_h} \mathcal{F}_f$ such that the diagram

$$
\begin{array}{ccc}
C_Y & \sim & (\mathcal{F}_g/Z) - \text{mod} \\
\downarrow h^\ast & & \downarrow \phi_h^\ast \\
C_X & \sim & (\mathcal{F}_f/Z) - \text{mod}
\end{array}
$$

quasi-commutes. Here $\phi_h^\ast$ is the inverse image functor associated with the monad morphism $\phi_h$ (a left adjoint to the pull-back functor). The pull-back functor, $\phi_{h^\ast}$, is, evidently, conservative and weakly continuous. The latter follows from the fact that the monads $\mathcal{F}_f$ and $\mathcal{F}_g$ are weakly continuous. $\blacksquare$

6. Continuous monads and affine morphisms. Duality. A functor $F$ is called continuous if it has a right adjoint. A monad $\mathcal{F} = (F, \mu)$ on $Y$ (i.e. on the category $C_Y$) is called continuous if the functor $F$ is continuous.

Dually, a comonad $\mathcal{G} = (G, \delta)$ on $Y$ is called cocontinuous if the functor $G$ has a left adjoint. In other words, a cocontinuous comonad on $Y$ is the same as a continuous monad on $Y^\circ$ and vice versa.

It follows that a continuous monad is weakly continuous, because the the functor $F$ preserves all colimits. Dually, a cocontinuous monad is weakly flat.

6.1. Duality. Let $\mathcal{F} = (F, \mu)$ be a continuous monad on $Y$; i.e. the functor $F$ has a right adjoint, $F^\wedge$. The multiplication $F^2 \xrightarrow{\mu} F$ induces a morphism $F^\wedge \xrightarrow{\delta} (F^\wedge)^2$ which is a comonad structure on $F^\wedge$ with the counit $F^\wedge \xrightarrow{\epsilon} \text{Id}_{C_Y}$ induced by the unit $\text{Id}_{C_Y} \xrightarrow{\eta} F$ of the monad $\mathcal{F}$. Thus, we have a comonad, $\mathcal{F}^\wedge = (F^\wedge, \mu^\wedge)$ dual to the monad $\mathcal{F}$. The map which assigns to any morphism $F^\wedge(L) \rightarrow L$, $L \in \text{Ob}C_Y$, the dual morphism $L \rightarrow F^\wedge(L)$ induces an isomorphism of categories

$$
\Phi : (\mathcal{F}/Y) - \text{mod} \xrightarrow{\sim} (Y\backslash \mathcal{F}^\wedge) - \text{Comod} \tag{1}
$$

such that the diagram

$$
(\mathcal{F}/Y) - \text{mod} \xrightarrow{\Phi} (Y\backslash \mathcal{F}^\wedge) - \text{Comod} \\
\downarrow \phi_* \quad \swarrow \phi^* \\
C_Y
$$

commutes. Here $\phi^*$ denotes the functor forgetting $\mathcal{F}^\wedge$-comodule structure.
It follows from the construction that $\mathcal{F}^\wedge$ is a cocontinuous comonad on $Y$ determined by the monad $\mathcal{F}$ uniquely up to isomorphism.

Conversely, to any cocontinuous comonad, $\mathcal{G} = (G, \delta)$, on $Y$, there corresponds a continuous monad $\mathcal{G}^\vee = (G^\vee, \delta^\vee)$, where $G^\vee$ is a left adjoint to $G$. The monad $\mathcal{G}^\vee$ is determined by $\mathcal{G}$ uniquely up to isomorphism, and we have a comonad and monad isomorphisms, respectively

$$\mathcal{G} \simto (\mathcal{G}^\vee)^\wedge \quad \text{and} \quad \mathcal{F} \simto (\mathcal{F}^\wedge)^\vee.$$ 

6.2. Proposition. A monad $\mathcal{F} = (F, \mu)$ on $Y$ is continuous iff the canonical morphism $\text{Sp}(\mathcal{F}/Y) \overset{\hat{f}}{\longrightarrow} Y$ is affine. Dually, a comonad $\mathcal{G} = (G, \delta)$ on $Y$ is cocontinuous iff the canonical morphism $Y \longrightarrow \text{Sp}^*(Y\setminus \mathcal{G})$ is coaffine.

Proof. A canonical direct image functor of $\hat{f}$ is the forgetful functor

$$(\mathcal{F}/Y) - \text{mod} \longrightarrow \mathcal{F}_Y, \quad (M, \xi) \longmapsto M.$$ 

Since the functor $\hat{f}_*$ is, evidently, conservative, the morphism $\hat{f}$ is affine iff $\hat{f}_*$ has a right adjoint.

(a) If $\hat{f}^!$ is a right adjoint to $\hat{f}_*$, then the functor $F^\wedge = \hat{f}_*\hat{f}^!$ is a right adjoint to $F = \hat{f}_*\hat{f}^*$. Here $\hat{f}^*$ denotes the functor $L \longrightarrow (F(L), \mu(L))$.

(b) Conversely, suppose $\mathcal{F} = (F, \mu)$ is a continuous monad on $Y$; i.e. the functor $F$ has a right adjoint, $F^\wedge$. The functor $\hat{f}^*$ in the diagram (2) has a right adjoint, $\hat{f}_*$, which maps every object $M$ of $\mathcal{F}_Y$ to the $(Y\setminus F^\wedge)$-comodule $(F^\wedge(M), M \overset{\delta(M)}{\longrightarrow} (F^\wedge)^2(M))$. It follows from the commutativity of (2) that the functor $\hat{f}^! = \Phi^{-1} \circ \hat{f}_*: \mathcal{F}_Y \longrightarrow \mathcal{F} - \text{mod}$ is a right adjoint to the forgetful functor $\mathcal{F} - \text{mod} \overset{\hat{f}_*}{\longrightarrow} \mathcal{F}_Y$. Since $\hat{f}_*$ is, obviously, conservative, it is a direct image functor of an affine morphism $\text{Sp}(\mathcal{F}/Y) \longrightarrow Y$. ■

6.2.1. Corollary. Suppose that the category $\mathcal{C}_Y$ has cokernels of reflexive pairs of arrows. A continuous morphism $X \overset{f}{\longrightarrow} Y$ in $\text{Cat}^\circ$ is affine iff its direct image functor $\mathcal{C}_X \overset{f_*}{\longrightarrow} \mathcal{C}_Y$ is the composition of a category equivalence

$$\mathcal{C}_X \longrightarrow (\mathcal{F}_f/Y) - \text{mod}$$

for a continuous monad $\mathcal{F}_f$ in $\mathcal{C}_Y$ and the forgetful functor $(\mathcal{F}_f/Y) - \text{mod} \longrightarrow \mathcal{C}_Y$. The monad $\mathcal{F}_f$ is determined by $f$ uniquely up to isomorphism.

Proof. The conditions of the Beck’s theorem are fulfilled if $f$ is affine, hence $f_*$ is the composition of an equivalence $\mathcal{C}_X \longrightarrow (\mathcal{F}_f/Y) - \text{mod}$ for a monad $\mathcal{F}_f = (f_*\hat{f}_*, \mu_f)$ in $\mathcal{C}_Y$ and the forgetful functor $(\mathcal{F}_f/Y) - \text{mod} \longrightarrow \mathcal{C}_Y$ (see (1)). The functor $F_f = f_*\hat{f}^*$ has a right adjoint $f_*\hat{f}^!$, where $\hat{f}^!$ is a right adjoint to $f_*$. The rest follows from 6.2. ■

6.3. Proposition. Suppose $X$ is an object of $\text{Cat}^\circ$ such that the category $\mathcal{C}_X$ has kernels of reflexive pairs of arrows. Let $\mathcal{F}_f = (F_f, \mu_f)$ and $\mathcal{F}_g = (F_g, \mu_g)$ be continuous monads on $X$. Then for any monad morphism $\mathcal{F}_f \overset{\varphi}{\longrightarrow} \mathcal{F}_g$, the corresponding morphism

$$\text{Sp}(\varphi): \text{Sp}(\mathcal{F}_f/X) \longrightarrow \text{Sp}(\mathcal{F}_g/X)$$

37
is affine.

Proof. The morphism $\varphi$ induces a dual comonad morphism $\mathcal{F}_g^\wedge \xrightarrow{\hat{\varphi}} \mathcal{F}_f^\wedge$ such that the diagram

$$(\mathcal{F}_g/X) - \text{mod} \xrightarrow{\varphi^*} (\mathcal{F}_f/X) - \text{mod}$$

$$\Phi_{\mathcal{F}_g} \downarrow \quad \downarrow \Phi_{\mathcal{F}_f}$$

$$(X\setminus \mathcal{F}_g^\wedge) - \text{Comod} \xrightarrow{\hat{\varphi}^*} (X\setminus \mathcal{F}_f^\wedge) - \text{Comod}$$

(3)

commutes. Here $\Phi_{\mathcal{F}_f}$ and $\Phi_{\mathcal{F}_g}$ are the canonical category isomorphisms (cf. 6.1). Since the category $C_X$ has kernels of coreflexive pairs of arrows and the functor $F_g^\wedge$ preserves limits (in particular, it preserves kernels of pairs of arrows), the functor $\hat{\varphi}^*$ has a right adjoint (cf. 5.3.2.2), hence $\varphi^*$ has a right adjoint. Since the functor $\varphi^*$ is conservative, the morphism $\text{Sp}(\varphi)$ is affine. ■

6.4. Proposition. Let

$$X \xrightarrow{h} Y$$

$$f \downarrow \quad \triangleleft \quad \check{g}$$

$$Z$$

be a commutative diagram in $|\text{Cat}|^o$. Suppose the category $C_Z$ has cokernels of coreflexive pairs of arrows. If $f$ and $g$ are affine, then $h$ is affine.

Proof. Fix inverse and direct image functors of $f$ and $g$ together with adjunction morphisms. By the Beck’s theorem, the canonical morphisms $C_X \longrightarrow (\mathcal{F}_f/Z) - \text{mod}$ and $C_Y \longrightarrow (\mathcal{F}_g/Z) - \text{mod}$ are category equivalences. Here $\mathcal{F}_f = (f_*f^*, \mu_f)$ and $\mathcal{F}_g = (g_*g^*, \mu_g)$ are monads associated with resp. $f$ and $g$. By 5.5.3 (or 10.7.1 in [R4]), a choice of an inverse image functor $h^*$ of the morphism $h$ determines a monad morphism $\mathcal{F}_g \xrightarrow{\phi_h} \mathcal{F}_f$ such that the diagram

$$C_Y \xrightarrow{h^*} (\mathcal{F}_g/Z) - \text{mod}$$

$$\downarrow \quad \downarrow \phi_h^*$$

$$C_X \xrightarrow{\sim} (\mathcal{F}_f/Z) - \text{mod}$$

quasi-commutes. By 6.3, since the monads $\mathcal{F}_g$ and $\mathcal{F}_f$ are continuous, the direct image functor $\phi_{h^*}$ (the pull-back by the morphism $\phi_h$) has a right adjoint, $\phi_h^!$. ■

For $Z \in \text{Ob}|\text{Cat}|^o$, denote by $\text{Aff}_Z$ the full subcategory of $|\text{Cat}|^o/Z$ whose objects are affine morphisms. Let $|\text{Cat}|^o_{\text{aff}}$ be the subcategory of $|\text{Cat}|^o$ formed by affine morphisms.

6.4.1. Proposition. Suppose that the category $C_Z$ has cokernels of reflexive pairs of arrows. Then the natural embedding $|\text{Cat}|^o_{\text{aff}}/Z \longrightarrow \text{Aff}_Z$ is an isomorphism of categories.

Proof. The assertion is a corollary of 6.4. ■

6.5. Proposition. Let $X \xrightarrow{f} Y$ be an affine morphism in $|\text{Cat}|^o$. If the category $C_Y$ is additive (resp. abelian, resp. abelian with small coproducts, resp. a Grothendieck category), then the category $C_X$ has the same property.
Proof. By 6.2, the category $C_X$ is equivalent to the category $(\mathcal{F}_f/Y)\mod$ of $(\mathcal{F}_f/Y)$-modules for a continuous monad $\mathcal{F} = (F_f, \mu_f)$ on $Y$. Since the functor $F_f$ has a right adjoint and the category $C_Y$ is additive, $F_f$ is additive and preserves colimits of arbitrary small diagrams. This implies that for any diagram $D \xrightarrow{\mathcal{D}} \mathcal{F} \mod$, the object $\text{colim}(f_\ast \circ \mathcal{D})$ (where $f_\ast$ is the forgetful functor $(\mathcal{F}/Y)\mod \to C_Y$) has a unique $(\mathcal{F}_f/Y)$-module structure $\xi_\mathcal{D}$ such that all morphisms $f_\ast \mathcal{D}(x) \to \text{colim}(f_\ast \circ \mathcal{D})$ are $(\mathcal{F}_f/Y)$-module morphisms $\mathcal{D}(x) \to (\text{colim}(f_\ast \circ \mathcal{D}), \xi_\mathcal{D})$. This implies the assertion. Details are left to the reader. ■

6.6. Affine morphisms to $\text{Sp}(R)$.

6.6.1. Proposition. Let $R$ be an associative unital ring. A continuous morphism $X \xrightarrow{f} \text{Sp}(R)$ in $|\text{Cat}|^o$ is affine iff its direct image functor, $C_X \xrightarrow{f_\ast} R\mod$, is the composition of an equivalence of categories $C_X \to R_f\mod$ for an associative unital ring $R_f$ and the pull-back functor $R_f\mod \xrightarrow{\phi_\ast} R\mod = C_Y$ for a ring morphism $R \xrightarrow{\phi} R_f$ determined by $f$ uniquely up to isomorphism.

Proof. (i) If $X \xrightarrow{f} Y$ is such a morphism in $|\text{Cat}|^o$ that $C_X \xrightarrow{f_\ast} C_Y$ is an equivalence of categories, then $f$ is, obviously, affine. The morphism $S\mod \to R\mod$ corresponding to a ring morphism $R \to S$ is affine by 3.8. Finally, the composition of affine morphisms is affine.

(ii) Conversely, suppose that $X \xrightarrow{f} \text{Sp}(R)$ is an affine morphism. Then the functor $f_\ast f^\ast : R\mod \to R\mod$ has a right adjoint, hence it is isomorphic to the functor $R_f \otimes_R - : L \to R_f \otimes_R L$ for some $R$-bimodule $R_f$. The monad structure on $f_\ast f^\ast$ induces an associative ring structure, $R_f \otimes_R R_f \xrightarrow{m_f} R_f$, on $R_f$; and the adjunction morphism $\text{Id}_{R\mod} \xrightarrow{\eta_f} f_\ast f^\ast$ corresponds to a ring morphism $R \xrightarrow{\phi} R_f$ so that the diagrams of functor morphisms

$$\begin{align*}
\text{Id}_{R\mod} & \xrightarrow{\sim} R \otimes_R - \\
\eta_f & \downarrow \quad \phi \otimes_R \\
f_\ast f^\ast & \xrightarrow{\sim} R_f \otimes_R -
\end{align*}$$

and

$$\begin{align*}
(f_\ast f^\ast)^2 & \xrightarrow{\sim} R_f \otimes_R R_f \otimes_R - \\
\mu_f & \downarrow m_f \\
f_\ast f^\ast & \xrightarrow{\sim} R_f \otimes_R -
\end{align*}$$

(2)

commute. Thus we have a commutative diagram

$$\begin{align*}
(\mathcal{F}_f/\text{Sp}(R))\mod & \xrightarrow{\sim} R_f\mod \\
\hat{f}_\ast & \downarrow \phi_\ast \\
R\mod & \xrightarrow{\phi_\ast}
\end{align*}$$

(3)

in which the horizontal arrow is an isomorphism of categories. Combining with the commutative diagram (1), we obtain a commutative diagram

$$\begin{align*}
C_X & \xrightarrow{\sim} R_f\mod \\
f_\ast & \downarrow \phi_\ast \\
R\mod & \xrightarrow{\phi_\ast}
\end{align*}$$

(4)

in which the horizontal arrow is an equivalence of categories.
Notice that $R_f = f_* f^*(R)$. Therefore, the ring morphism $R \xrightarrow{\phi} R_f$ is defined uniquely up to isomorphism by a choice of the functor $f^*$. □

6.6.2. A comparison of two descriptions. Let $X \xrightarrow{f} \text{Sp}(R)$ be an affine morphism. Being continuous, the morphism $f$ is determined uniquely up to isomorphism by the object $\mathcal{O} = f^*(R)$, and a right $R$-module structure $R \rightarrow \Gamma_X \mathcal{O} = C_X(\mathcal{O}, \mathcal{O})$ (cf. 4.1). By 4.5, we have a commutative diagram of direct image functors of continuous morphisms

\[
\begin{array}{ccc}
C_X & \xrightarrow{f_*} & \Gamma_X \mathcal{O} \text{-mod} \\
\downarrow f_\ast & \swarrow \phi_f \ast & \\
R \text{-mod}
\end{array}
\]

(1)

Here $\phi_f \ast$ is the pull-back by the ring morphism $R \xrightarrow{\phi_f} \Gamma_X \mathcal{O}$ defining a right $R$-module structure on $\mathcal{O}$. The morphism $f_\ast$ has an inverse image functor $f^* \mathcal{O}$ which maps the left module $\mathcal{O}$ to $\mathcal{O}$. The adjunction morphism $\Gamma_X \mathcal{O} \rightarrow f_\ast f^* \mathcal{O}(\Gamma_X \mathcal{O})$ is an isomorphism.

Since morphisms $f$ and $\text{Sp}(\Gamma_X \mathcal{O}) \xrightarrow{\phi_f} \text{Sp}(R)$ are affine, the morphism $f_\ast$ is affine too (cf. 6.3). In particular, $f_\ast f^* \mathcal{O}$ has a right adjoint, hence it preserves colimits. Since $\Gamma_X \mathcal{O}$ is a generator of the category $\Gamma_X \mathcal{O} \text{-mod}$, the isomorphism of the adjunction arrow $\Gamma_X \mathcal{O} \rightarrow f_\ast f^* \mathcal{O}(\Gamma_X \mathcal{O})$ implies that $M \rightarrow f_\ast f^* \mathcal{O}(M)$ is an isomorphism for any $\Gamma_X \mathcal{O}$-module $M$. This means that the functor $f^* \mathcal{O}$ is fully faithful, hence $f_\ast$ is a localization. Since by condition $f_\ast$ is conservative, it is a category equivalence.

This shows that $C_X$ is naturally equivalent to the category of $\Gamma_X \mathcal{O}$-modules. Thus the ring morphism $R \rightarrow R_f$ in 6.6.1 is isomorphic to the ring morphism $R \rightarrow \Gamma_X \mathcal{O}$ defining a right $R$-module structure on the object $\mathcal{O}$.

This observation was made before (in [R], Ch.7) using a slightly different argument.

7. Flat descent.

7.1. Continuous, flat comonads. A comonad $\mathcal{G} = (G, \delta)$ on $X$ is called
— continuous if the functor $G$ has a right adjoint,
— flat if the functor $G$ preserves finite limits,
— weakly flat if the functor $G$ preserves kernels of coreflexive pairs of arrows,
— conservative if the functor $G$ is conservative.

7.2. Proposition. Let $X \xrightarrow{f} Y$ be a continuous morphism, and let $C_X$ have kernels of coreflexive pairs of morphisms. The morphism $X \xrightarrow{f} Y$ in $|\text{Cat}|^o$ is affine, flat (resp. weakly flat), and conservative iff its inverse image functor $C_Y \xrightarrow{f^*} C_X$ is the composition of an equivalence $C_Y \rightarrow (X \setminus \mathcal{G}_f) \text{-Comod}$ for a continuous flat (resp. weakly flat) conservative comonad $\mathcal{G}_f$ on $X$ and the forgetful functor $(X \setminus \mathcal{G}_f) \text{-Comod} \rightarrow C_X$. The monad $\mathcal{G}_f$ is determined by $f$ uniquely up to isomorphism.

Proof. The conditions of the Beck’s theorem are fulfilled if $f$ is weakly flat and conservative, hence $f^*$ is the composition of an equivalence $C_Y \rightarrow (X \setminus \mathcal{G}_f) \text{-Comod}$ for a comonad $\mathcal{G}_f = (f^* f_\ast, \delta_f)$ on $X$ and the forgetful functor $(X \setminus \mathcal{G}_f) \text{-Comod} \rightarrow C_X$ (see
If $f$ is affine, then the functor $G_f = f^* f_*$ has a right adjoint $f^! f_*$, where $f^!$ is a right adjoint to $f_*$. Let now $\mathcal{G} = (G, \delta)$ be a continuous comonad on $X$ and $f^*$ the forgetful functor $(X\setminus \mathcal{G}) - \text{Comod} \longrightarrow C_X$. The functor $f_*$ which assigns to each object $M$ of $C_X$ the $\mathcal{G}$-comodule $(G(M), \delta(M))$ is right adjoint to $f^*$: the canonical adjunction arrows are
\[
\epsilon_f : f^* f_* = G \longrightarrow \text{Id}_{C_X} \quad \text{and} \quad \eta_f : \text{Id}_{C_Y} \longrightarrow f_* f^*,
\]
where $C_Y = \mathcal{G} - \text{Comod}$, $\epsilon_f$ is the counit of the monad $\mathcal{G}$ and
\[
\eta_f(M, \xi) = \xi : (M, \xi) \longrightarrow f_* f^*(M, \xi) = (G(M), \delta(M))
\]
for any $\mathcal{G}$-comodule $(M, M \xrightarrow{\xi} G(M))$.

Let $G^!$ be a right adjoint to $G$, and let $G^! G^! \xrightarrow{\epsilon^!} \text{Id}_{C_X}$ and $\text{Id}_{C_X} \xrightarrow{\eta^!} G^! G$ be adjunction arrows. Let $f^!$ denote the functor
\[
C_Y = (X\setminus \mathcal{G}) - \text{Comod} \longrightarrow C_X, \ (M, \xi) \mapsto G^!(M).
\]
Since $f^! f_* = G^! G$, the adjunction arrow $\eta^!$ is a morphism $\text{Id}_{C_X} \longrightarrow f^! f_*$. The composition $f_* f^!$ assigns to each $\mathcal{G}$-comodule $(M, \xi)$ the $\mathcal{G}$-comodule $f_* G^!(M) = (G G^!(M), \delta G^!(M))$. One can check that the adjunction arrow $\epsilon^!(M) : G G^!(M) \longrightarrow M$ is a $\mathcal{G}$-comodule morphism $f_* G^!(M) \longrightarrow (M, \xi)$, i.e. the diagram
\[
\begin{array}{ccc}
G G^!(M) & \xrightarrow{\epsilon^!(M)} & M \\
\delta G^!(M) \downarrow & & \downarrow \xi \\
G^2 G^!(M) & \xrightarrow{G \epsilon^!(M)} & G(M)
\end{array}
\]
commutes. This implies that $\epsilon^! f^*$ and $\eta^!$ are adjunction morphisms, hence the assertion. \hfill \blacksquare

**7.2.1. Corollary.** Let a morphism $X \xrightarrow{f} Y$ be affine, weakly flat, and conservative. If the category $C_X$ is additive (resp. abelian, resp. abelian with small coproducts, resp. a Grothendieck category), then the category $C_Y$ has the same property, and the morphism $f$ is flat.

**Proof.** Under the hypothesis, the category $C_Y$ is equivalent to the category $(X\setminus \mathcal{G}_f)$-comodules for a continuous comonad $\mathcal{G}_f = (G_f, \delta_f)$ on $X$. Since the functor $C_X \xrightarrow{G_f} C_X$ has a right adjoint, it is additive and preserves small colimits. Since $G_f = f^* f_*$, the functor $f_*$ preserves all small limits, and the functor $f^*$ preserves kernels of coreflexive pairs of arrows, the functor $G_f$ preserves kernels of coreflexive pairs of arrows too. For additive categories (more generally, for categories with coproducts and a zero object) functors which preserve kernels of coreflexive pairs of arrows preserve kernels of any pairs of arrows. Thus $G_f$ preserves kernels of any pairs of arrows and, being additive, products (which coincide with coproducts), hence $G_f$ reserves limits of any finite diagrams. This implies that the category $(X\setminus \mathcal{G}_f) - \text{Comod}$ has limits of finite diagrams which are preserved (and
reflected) by the forgetful functor \((X \setminus \mathcal{G}_f) - \text{Comod} \to C_X\). This implies the additivity of \((X \setminus \mathcal{G}_f) - \text{Comod}\). The rest follows from the compatibility of \(G_f\) with arbitrary small colimits (cf. the argument of 6.5).

7.3. Affine, flat morphisms from \(\text{Sp}(R)\). If \(R\) is an associative ring and \(\mathcal{G}\) a comonad on \(\text{Sp}(R)\), we shall write for convenience \((R \setminus \mathcal{G})\) instead of \((\text{Sp}(R) \setminus \mathcal{G})\).

7.3.1. Proposition. A continuous morphism \(\text{Sp}(R) \xrightarrow{f} X\) in \(|\text{Cat}|^o\) is flat, conservative, and affine iff its inverse image functor, \(C_X \xrightarrow{f^*} R - \text{mod}\), is the composition of an equivalence of categories \(C_X \to (R \setminus \mathcal{H}_f) - \text{Comod}\) for a coalgebra \(\mathcal{H}_f = (H_f, \delta_f)\) in the category of \(R\)-bimodules such that \(H_f\) is a flat right \(R\)-module, and the forgetful functor \((R \setminus \mathcal{H}_f) - \text{mod} \to R - \text{mod}\).

Proof. Let \(\text{Sp}(R) \xrightarrow{f} X\) be a flat, conservative, and affine morphism with an inverse image functor \(f^*\). By 7.2, the functor \(C_X \xrightarrow{f^*} R - \text{mod}\) is the composition of a category equivalence \(C_X \to (R \setminus \mathcal{G}_f) - \text{Comod}\) for a comonad \(\mathcal{G}_f = (G_f, \delta_f)\) on \(\text{Sp}(R)\) and the forgetful functor \((R \setminus \mathcal{G}_f) - \text{mod} \to R - \text{mod}\). Since the comonad \(\mathcal{G}_f\) is continuous, the functor \(G_f\) is isomorphic to the functor \(H_f \otimes_R -\) for an \(R\)-bimodule \(H_f\) (equal to \(G_f(R)\)). The comultiplication \(G_f \xrightarrow{\delta_f} G_f^2\) induces a comultiplication \(H_f \to H_f \otimes_R G_f\) (see the argument of 6.6.1).

Conversely, let \(\mathcal{H} = (H, \delta)\) be a coalgebra in the category of \(R\)-bimodules, and let \(f^*\) denote the forgetful functor
\[(R \setminus \mathcal{H}) - \text{Comod} \to R - \text{mod}, \quad (M, M \to H(M)) \mapsto M.\] (1)
The functor \(f^*\) has a right adjoint,
\[L \mapsfrom \mathcal{H} \otimes_R L = (H \otimes_R L, \delta \otimes_R L)\] (2)
(see the argument of 7.2). The comonad \(\mathcal{H} \otimes_R\) is continuous, since the functor \(H \otimes_R -\) has a right adjoint, \(\text{Hom}_R(H, -)\).

The functor \(f^*\) being flat is equivalent to the flatness of \(H\) as a right \(R\)-module. The assertion follows now from 7.2. ■

7.3.2. Corollary. Let \((C, \mathcal{O})\) be a ringed category, and let \(X \xrightarrow{f} Y\) be a morphism of presheaves of sets on \(C\) such that its inverse image functor, \(Q\text{coh}_Y \xrightarrow{f^*} Q\text{coh}_X\), is flat, conservative, and its direct image functor is conservative and has a right adjoint. Suppose \(X\) is representable. Then there exists a coalgebra \(\mathcal{H}\) in the category of \(\mathcal{O}(X)\)-bimodules such that \(f^*\) is the composition of a category equivalence \(Q\text{coh}_Y \xrightarrow{f^*} (\mathcal{O}(X) \setminus \mathcal{H}) - \text{Comod}\) and the forgetful functor \((\mathcal{O}(X) \setminus \mathcal{H}) - \text{Comod} \to \mathcal{O}(X) - \text{mod}\).

Proof. The assertion follows from 7.3.1. ■

7.3.3. Example: semiseparated schemes and algebraic spaces. Let \(\mathcal{X}\) be a scheme, or an algebraic space. Recall that an affine cover \(\{U_i \xrightarrow{u_i} \mathcal{X} \mid i \in J\}\) of \(\mathcal{X}\) is called semiseparated if each morphism \(U_i \xrightarrow{u_i} \mathcal{X}\) is affine. A scheme (or an algebraic space) is
called \emph{semiseparated} if it has a semiseparated cover. Evidently, every separated algebraic space (or scheme) is semiseparated.

If \( \{U_i \to X \mid i \in J\} \) is a semiseparated cover of \( X \), the corresponding morphism

\[
\mathcal{U} = \prod_{i \in J} U_i \xrightarrow{\pi} X
\]

is affine which implies that the space of relations, \( \mathcal{R} = \prod_{i,j \in J} U_i \times_X U_j \simeq \mathcal{U} \times_X \mathcal{U} \) is affine too. Since morphisms \( u_i \) are étale, their inverse image functors, \( u_i^* \) are flat and the family \( \{u_i^* \mid i \in J\} \) is conservative. The latter means exactly that an inverse image functor \( \pi^* \) of the morphism \( \pi \) is flat and conservative. It follows by construction, that the inverse images of projections \( \mathcal{R} \Rightarrow \mathcal{U} \) are flat and conservative (equivalently, faithfully flat). And they are affine, since both \( \mathcal{R} \) and \( \mathcal{U} \) are affine.

8. Generalities on finiteness conditions and smooth and étale morphisms.

Fix a functor \( \mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B} \). In what follows, \( \mathfrak{A} \) is a category of ‘local’, or ‘affine’, objects, \( \mathfrak{B} \) is a category of spaces, and \( \mathfrak{F} \) assigns to local objects corresponding spaces. In a standard commutative prototype, \( \mathfrak{B} \) is the category of locally ringed topological spaces, otherwise called \emph{geometric spaces}, \( \mathfrak{A} \) is the category opposite to the category \( CRings \) of commutative unital rings, and the functor \( \mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B} \) assigns to every commutative ring \( R \) the affine scheme \((\text{Spec}R, \mathcal{O}_R)\).

8.1. Locally finitely presentable objects and morphisms. Given a functor \( \mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B} \), we call an object \( X \) of \( \mathfrak{B} \) of \((\mathfrak{A}, \mathfrak{F})\)-\emph{finite type} (resp. \((\mathfrak{A}, \mathfrak{F})\)-\emph{finitely presentable}), if for any filtered projective system \( D \xrightarrow{\mathfrak{D}} \mathfrak{A} \) such that there exists \( \text{lim}(\mathfrak{F} \circ \mathfrak{D}) \), the canonical map

\[
\text{colim} \mathfrak{B}(\mathfrak{F} \circ \mathfrak{D}, X) \longrightarrow \mathfrak{B}(\text{lim}(\mathfrak{F} \circ \mathfrak{D}), X)
\]

is injective (resp. bijective).

We call a morphism \( X \xrightarrow{f} Y \) of \( \mathfrak{B} \) of \((\mathfrak{A}, \mathfrak{F})\)-\emph{finite type} (resp. \((\mathfrak{A}, \mathfrak{F})\)-\emph{finitely presentable}), if for any filtered projective system \( D \xrightarrow{\mathfrak{D}} \mathfrak{A}/Y \), the canonical morphism

\[
\text{colim} \mathfrak{B}/Y(\mathfrak{F}_Y \circ \mathfrak{D}, (X, f)) \longrightarrow \mathfrak{B}/Y(\text{lim}(\mathfrak{F}_Y \circ \mathfrak{D}), (X, f))
\]

is injective (resp. bijective), provided \( \text{lim}(\mathfrak{F}_Y \circ \mathfrak{D}) \) exists. Here \( \mathfrak{F}_Y \) denotes the canonical functor \( \mathfrak{F}/Y \). It follows from these definitions that if the category \( \mathfrak{B} \) has a final object, \( \bullet \), then an object \( X \) of \( \mathfrak{B} \) is of \((\mathfrak{A}, \mathfrak{F})\)-finite type (resp. \( \mathfrak{A} \)-finitely presentable) iﬀ the unique morphism \( X \longrightarrow \bullet \) is of \((\mathfrak{A}, \mathfrak{F})\)-finite type (resp. \((\mathfrak{A}, \mathfrak{F})\)-finitely presentable).

8.1.1. Proposition. Let \( \Sigma^1_{\mathfrak{A}} \) (resp. \( \Sigma^0_{\mathfrak{A}} \)) denote the class of all \((\mathfrak{A}, \mathfrak{F})\)-finitely presentable morphisms (resp. morphisms of \((\mathfrak{A}, \mathfrak{F})\)-finite type) of the category \( \mathfrak{B} \).

(a) Both \( \Sigma^0_{\mathfrak{A}} \) and \( \Sigma^1_{\mathfrak{A}} \) are closed under compositions and contain all isomorphisms.
(b) If the morphism \( f \) in the cartesian square

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

belongs to \( \Sigma^i_\mathcal{A} \), then \( f' \) belongs to \( \Sigma^i_\mathcal{A} \), \( i = 0, 1 \).

(c) Suppose that \( X \xrightarrow{f} Y \) and \( Z \xrightarrow{h} W \) are morphisms over an object \( S \) which belong to \( \Sigma^i_\mathcal{A} \). If \( X \times_S Z \) and \( Y \times_S W \) exist, then the morphism \( X \times_S Z \xrightarrow{f \times_S h} Y \times_S W \) belongs to \( \Sigma^i_\mathcal{A} \), \( i = 0, 1 \).

(d) If the composition \( g \circ f \) of two morphisms is \((\mathcal{A}, \mathcal{F})\)-finitely presentable and \( g \) is of \((\mathcal{A}, \mathcal{F})\)-finite type, then \( f \) is \((\mathcal{A}, \mathcal{F})\)-finitely presentable.

Proof. The argument is similar to that of [KR2, 5.12.2].

8.2. Formally smooth, formally unramified and formally étale morphisms.

Fix a functor \( \mathcal{A} \xrightarrow{\mathcal{F}} \mathcal{B} \) and a family, \( \mathcal{M} \), of morphisms of \( \mathcal{A} \) containing all identical morphisms.

8.2.1. Definitions. (i) We call a morphism \( X \xrightarrow{f} Y \) of \( \mathcal{B} \) formally \( \mathcal{M} \)-smooth if any commutative diagram

\[
\begin{array}{ccc}
\mathcal{F}(T) & \xrightarrow{g} & X \\
\downarrow \mathcal{F}(\phi) & & \downarrow f \\
\mathcal{F}(S) & \xrightarrow{g'} & Y
\end{array}
\] (1)

such that \( \phi \in \mathcal{M} \) extends to a commutative diagram

\[
\begin{array}{ccc}
\mathcal{F}(T) & \xrightarrow{g'} & X \\
\downarrow \mathcal{F}(\phi) & \nearrow \gamma & \downarrow f \\
\mathcal{F}(S) & \xrightarrow{g} & Y
\end{array}
\] (2)

(ii) We call \( X \xrightarrow{f} Y \) formally \( \mathcal{M} \)-unramified if for any commutative diagram (1) such that \( \phi \in \mathcal{M} \), there exists at most one morphism \( S \xrightarrow{\gamma} X \) such that the diagram (2) commutes.

(iii) We call \( X \xrightarrow{f} Y \) formally \( \mathcal{M} \)-étale if it is both formally \( \mathcal{M} \)-smooth and formally \( \mathcal{M} \)-unramified.

We denote by \( \mathcal{M}_{fsm} \) (resp. \( \mathcal{M}_{fnr} \), resp. \( \mathcal{M}_{fet} \)) the class of all formally \( \mathcal{M} \)-smooth (resp. formally \( \mathcal{M} \)-unramified, resp. formally \( \mathcal{M} \)-étale) morphisms.

8.2.2. Proposition. (a) Each monomorphism is formally unramified and each isomorphism is formally étale.
(b) Composition of formally $\mathcal{M}$-smooth (resp. formally $\mathcal{M}$-unramified, resp. formally $\mathcal{M}$-étale) morphisms is formally $\mathcal{M}$-smooth (resp. formally $\mathcal{M}$-unramified, resp. formally $\mathcal{M}$-étale).

(c) Let $X \xrightarrow{f} Y$, $Y \xrightarrow{h} Z$ be morphisms of $\mathcal{B}$.
(i) If $h \circ f$ is formally $\mathcal{M}$-unramified, then $f$ is formally $\mathcal{M}$-unramified.
(ii) Suppose $h$ is formally $\mathcal{M}$-unramified. If $X \xrightarrow{h \circ f} Z$ is formally $\mathcal{M}$-smooth (resp. formally $\mathcal{M}$-étale), then $f$ is formally $\mathcal{M}$-smooth (resp. formally $\mathcal{M}$-étale).

(d) Let $X \xrightarrow{\xi} S \xleftarrow{\xi'} X'$ and $Y \xrightarrow{\nu} S \xleftarrow{\nu'} Y'$ be morphisms such that there exist $X \times_{S} X'$ and $Y \times_{S} Y'$. Let $(X, \xi) \xrightarrow{f} (Y, \nu)$ and $(X', \xi') \xrightarrow{f'} (Y', \nu)$ be morphisms of objects over $S$. The morphisms $f$, $f'$ are formally $\mathcal{M}$-smooth (resp. formally $\mathcal{M}$-unramified, resp. formally $\mathcal{M}$-étale) iff the morphism $f \times_{S} f' : X \times_{S} X' \longrightarrow Y \times_{S} Y'$ has the respective property.

(e) Let $X \xrightarrow{f} S \xleftarrow{h} Y$ be such a diagram that there exists a fiber product $X \times_{S} Y$. If $f$ is formally $\mathcal{M}$-smooth (resp. formally $\mathcal{M}$-unramified, resp. formally étale), then the canonical projection $X \times_{S} Y \xrightarrow{f'} Y$ is formally $\mathcal{M}$-smooth (resp. formally $\mathcal{M}$-unramified, resp. formally $\mathcal{M}$-étale).

Proof. See [KR2, 6.6].

8.3. Smooth, unramified, and étale morphisms. We call a morphism $X \xrightarrow{f} Y$ $\mathcal{M}$-smooth (resp. $\mathcal{M}$-étale, resp. $\mathcal{M}$-unramified) if it is $(\mathcal{A}, \mathfrak{F})$-finitely presentable and formally $\mathcal{M}$-smooth (resp. formally $\mathcal{M}$-unramified, resp. formally $\mathcal{M}$-étale).

We denote by $\mathcal{M}_{sm}$ (resp. $\mathcal{M}_{nr}$, resp. $\mathcal{M}_{et}$) the family of all $\mathcal{M}$-smooth (resp. $\mathcal{M}$-unramified, resp. $\mathcal{M}$-étale) morphisms.

We call a morphism $X \xrightarrow{f} Y$ $\mathcal{M}$-open immersion if it is an $\mathcal{M}$-smooth monomorphism.

8.3.1. Proposition. (a) Each monomorphism is $\mathcal{M}$-unramified and each isomorphism is $\mathcal{M}$-open immersion.

(b) Composition of $\mathcal{M}$-smooth (resp. $\mathcal{M}$-unramified, resp. $\mathcal{M}$-étale) morphisms is $\mathcal{M}$-smooth (resp. $\mathcal{M}$-unramified, resp. $\mathcal{M}$-étale).

(c) Let $X \xrightarrow{f} Y$, $Y \xrightarrow{h} Z$ be morphisms of $\mathcal{B}$.
(i) If $g \circ f$ is formally $\mathcal{M}$-unramified and $g$ is of $\mathcal{M}$-finite type, then $f$ is $\mathcal{M}$-unramified.
(ii) Suppose $g$ is $\mathcal{M}$-unramified. If $X \xrightarrow{g \circ f} Z$ is $\mathcal{M}$-smooth (resp. $\mathcal{M}$-étale), then $f$ is $\mathcal{M}$-smooth (resp. $\mathcal{M}$-étale).

(d) Let $X \xrightarrow{\xi} S \xleftarrow{\xi'} X'$ and $Y \xrightarrow{\nu} S \xleftarrow{\nu'} Y'$ be morphisms such that there exist $X \times_{S} X'$ and $Y \times_{S} Y'$. Let $(X, \xi) \xrightarrow{f} (Y, \nu)$ and $(X', \xi') \xrightarrow{f'} (Y', \nu)$ be morphisms of objects over $S$. The morphisms $f$, $f'$ are $\mathcal{M}$-smooth (resp. $\mathcal{M}$-unramified, resp. $\mathcal{M}$-étale) iff $f \times_{S} f' : X \times_{S} X' \longrightarrow Y \times_{S} Y'$ has the respective property.

(e) Let $X \xrightarrow{f} S \xleftarrow{h} Y$ be such a diagram that there exists a fiber product $X \times_{S} Y$. If $f$ is $\mathcal{M}$-smooth (resp. $\mathcal{M}$-unramified, resp. étale), then the projection $X \times_{S} Y \xrightarrow{f'} Y$ is $\mathcal{M}$-smooth (resp. $\mathcal{M}$-unramified, resp. $\mathcal{M}$-étale).
8.4. Standard examples.

8.4.1. Let \( \mathfrak{A} \) be the category \( CRings^{op} \), as in 8.4.1. Let \( \mathfrak{B} \) be the category \( \mathcal{E}sp \) of spaces in the sense of Grothendieck (and \([DG]\)); i.e. \( \mathfrak{B} \) is the category of sheaves of sets on \( \mathfrak{A} \) for the flat topology. In other words, objects of \( \mathfrak{B} \) are functors \( CRings \to Sets \) which preserve finite products, and for any faithfully flat ring morphism \( R \to T \), the diagram

\[
\begin{array}{ccc}
X(R) & \longrightarrow & X(T) \\
\longrightarrow & & \longrightarrow \\
& X(T \otimes_R T)
\end{array}
\]

is exact. The functor \( \mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B} \) is the Yoneda functor which maps every object \( R \) of \( \mathfrak{A} \) to the functor \( \mathfrak{A}(-, R) = CRings(R, -) \) represented by \( R \) (here we identify objects of \( \mathfrak{A} \) with the corresponding objects of \( CRings \)).

Then \( (\mathfrak{A}, \mathfrak{F}) \)-finitely presentable morphisms (resp. morphisms of \( (\mathfrak{A}, \mathfrak{F}) \)-finite type) are precisely locally finitely presentable morphisms (resp. morphisms of locally finite type).

We take as \( \mathcal{M} \) the family of all morphisms of \( \mathfrak{A} \) such that the corresponding ring morphism is an epimorphism with a nilpotent kernel. Then formally \( \mathcal{M} \)-smooth (resp. formally \( \mathcal{M} \)-unramified, resp. formally \( \mathcal{M} \)-étale) morphisms are formally smooth (resp. formally unramified, resp. formally étale) in the usual sense. Therefore, \( \mathcal{M} \)-smooth, \( \mathcal{M} \)-unramified, \( \mathcal{M} \)-étale morphisms are resp. smooth, unramified and étale. And \( \mathcal{M} \)-open immersions are precisely open immersions in the conventional sense.

8.4.2. Let \( \mathfrak{A} \) be the opposite category to the category \( Alg_k \) of associative unital \( k \)-algebras, Let \( \mathfrak{B} \) be the category of presheaves of sets on \( \mathfrak{A} \), i.e. functors \( Alg_k \to Sets \), which are local in the following sense: they preserve finite products, and for any faithfully flat \( k \)-algebra morphism \( R \to T \), the diagram

\[
\begin{array}{ccc}
X(R) & \longrightarrow & X(T) \\
\longrightarrow & & \longrightarrow \\
& X(T \star_R T)
\end{array}
\]

is exact. Here \( \star_R \) denote the 'star'-product of rings over \( R \) (which is a traditional name for a push-forward of associative rings). We denote this category by \( \mathcal{E}sp_{NC} \) and call its objects 'noncommutative spaces', or simply 'spaces'. The functor \( \mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B} \) is the Yoneda embedding, \( R \mapsto \mathfrak{A}(-, R) = Alg_k(R, -) \) (here we identify objects of \( \mathfrak{A} \) with the corresponding \( k \)-algebras).

It follows that \( (\mathfrak{A}, \mathfrak{F}) \)-finitely presentable morphisms (resp. morphisms of \( (\mathfrak{A}, \mathfrak{F}) \)-finite type) are precisely locally finitely presentable morphisms (resp. morphisms of locally finite type) in the sense of [KR, 5.12.1].

We take as \( \mathcal{M} \) the family of all morphisms of \( \mathfrak{A} \) such that the corresponding \( k \)-algebra morphism is an epimorphism with a nilpotent kernel. Then formally \( \mathcal{M} \)-smooth (resp. formally \( \mathcal{M} \)-unramified, resp. formally \( \mathcal{M} \)-étale) morphisms are formally smooth (resp. formally unramified, resp. formally étale) in the sense of [KR2, 5.8].

8.5. Smooth and étale morphisms of 'spaces'. Open immersions.

8.5.1. Cosubspaces and closed immersions. Let \( X \) be a 'space', that is an object of \(|Cat|^{op}\). We call \( Y \) a cosubspace of \( X \) if \( C_Y \) is a full subcategory of the category \( C_X \).
which is closed under finite limits and colimits taken in $C_X$ and, in addition, the following condition holds:

If $M \rightarrow N$ is a pair of arrows such that $M \in ObC_Y$ (resp. $N \in ObC_Y$) and the kernel (resp. cokernel) of the pair $M \rightarrow N$ exists in $C_X$, then this kernel (resp. cokernel) belongs to the subcategory $C_Y$. In particular, $C_Y$ is strictly full (i.e. it contains with any of its objects all objects of $C_X$ isomorphic to this object).

We call a continuous morphism $U \xrightarrow{u} X$ a closed immersion if its direct image functor, $u_*$, induces an equivalence between $C_U$ and $C_Y$ for some cosubspace $Y$ of $X$. In particular, $u_*$ is a fully faithful.

8.5.1.1. Topologizing subcategories. Suppose $C_X$ is an abelian category, and $C_Y$ is a subcategory of $C_X$. Then $Y$ is a cosubspace of $X$ iff $C_Y$ is a topologizing subcategory; i.e. $C_Y$ is closed by direct sums and subquotients (taken in $C_X$).

8.5.2. Thickennings. We call a closed immersion $U \xrightarrow{u} T$ a thickening (and say that $T$ is a thickening of $U$), if the smallest saturated multiplicative system in $HomC_T$ containing $u_*(HomC_U)$ coincides with $HomC_T$. The latter condition means precisely that if $Y \xrightarrow{q} X$ is an exact localization such that the composition $q^*u_*$ maps all arrows to isomorphisms, then $C_Y$ is a groupoid.

8.5.2.1. Abelian case. Suppose that the category $C_T$ is abelian. Then a continuous morphism $U \xrightarrow{u} T$ is a thickening iff its direct image functor, $u_*$, induces an equivalence between $C_U$ and a topologizing subcategory of $C_T$, and the smallest thick subcategory of $C_T$ containing $u_*(C_U)$ coincides with $C_T$.

8.5.3. Formally smooth, formally unramified, and formally ´etale morphisms. Fix a ‘space’ $S$, and consider the category $|Cat|_S^o$ and its full subcategory $Aff_S$. Recall that $|Cat|_S^o$ is a full subcategory of $|Cat|^o/S$ whose objects are pair $(X,f)$, where $X \xrightarrow{f} S$ is a continuous morphism; and $Aff_S$ is the full subcategory of $|Cat|^o/S$ whose objects are pairs $(Y,g)$, where $Y \xrightarrow{g} S$ is an affine morphism.

Let $X \xrightarrow{f} Y$ be a morphism in $|Cat|^o/S$. We call the morphism $f$ formally smooth (resp. formally unramified) if for every commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{g'} & X \\
\downarrow u & & \downarrow f \\
T & \xrightarrow{g} & Y
\end{array}
$$

of $S$-‘spaces’ such that $U$ and $T$ belong to $Aff_S$ and $U \xrightarrow{u} T$ is a thickening, there exists (resp. at most one) morphism $T \xrightarrow{\gamma} X$ such that the diagram

$$
\begin{array}{ccc}
U & \xrightarrow{g'} & X \\
\downarrow u & & \downarrow f \\
T & \xrightarrow{g} & Y
\end{array}
$$

47
commutes. The morphism \( X \xrightarrow{f} Y \) is called formally étale if it is both formally smooth and formally unramified.

We call an \( S \)-space \((X, \xi)\) formally smooth (resp. formally unramified, resp. formally étale), if the morphism \((X, \xi) \xrightarrow{\xi} (S, id_S)\) is formally smooth (resp. formally unramified, resp. formally étale).

8.5.3.1. Proposition. (a) The class of formally smooth (resp. formally unramified, resp. formally étale) morphisms is closed under composition and contains all isomorphisms.

(b) If \( h \circ f \) is formally unramified, then \( f \) is formally unramified.

(c) Suppose a morphism \( h \) is formally unramified. If \( X \xrightarrow{h \circ f} Z \) is formally smooth (resp. formally étale), then \( f \) is formally smooth (resp. formally étale).

(d) Let \( X \xleftarrow{\xi} T \xleftarrow{\xi'} X' \) and \( Y \xrightarrow{\nu} T \xleftarrow{\nu'} Y' \) be morphisms such that there exist \( X \times_T X' \) and \( Y \times_T Y' \). Let \((X, \xi) \xrightarrow{f} (Y, \nu)\) and \((X', \xi') \xrightarrow{f'} (Y', \nu)\) be morphisms of objects over \( T \). The morphisms \( f, f' \) are formally smooth (resp. formally unramified, resp. formally étale) iff the morphism \( f \times_T f' : X \times_T X' \rightarrow Y \times_T Y' \) has the respective property.

(e) Let \( X \xrightarrow{f} T \xleftarrow{h} Y \) be such a diagram that there exists a fiber product \( X \times_T Y \). If \( f \) is formally smooth (resp. formally unramified, resp. formally étale), then the canonical projection \( X \times_T Y \xrightarrow{f'} Y \) is formally smooth (resp. formally unramified, resp. formally étale).

Proof. The assertion follows from [KR2 6.6].

8.5.5. Locally finitely presentable \( S \)-spaces and morphisms of \( S \)-spaces. Let \( S \) be a \( S \)-space. We take as \( B \) the category \(|\text{Cat}_o S|\). If \( A \) is the subcategory \( \text{Aff}_S \) of affine \( S \)-spaces, then we shall call \( A \)-presentable \( S \)-spaces (resp. morphisms of \( S \)-spaces) locally presentable. Similarly, we call (morphisms of) \( S \)-spaces of \( A \)-finite type locally of finite type.

8.5.6. Smooth, unramified, and étale \( S \)-spaces and morphisms of \( S \)-spaces. Open immersions. We call a morphism \( X \xrightarrow{f} Y \) of \( S \)-spaces smooth (resp. étale, resp. unramified) if it is locally finitely presentable and formally smooth (resp. formally étale, resp. formally unramified).

We call a morphism \( X \xrightarrow{f} Y \) of \( S \)-spaces an open immersion if it is a smooth monomorphism.

An \( S \)-space \((X, \xi)\) is called smooth (resp. unramified, resp. étale) iff it is formally smooth (resp. formally unramified, resp. formally étale) and locally finitely presentable.

Since \((S, id_S)\) is a final object in \(|\text{Cat}_o S|\), an \( S \)-space \((X, \xi)\) is smooth (resp. unramified, resp. étale) if the unique morphism \((X, \xi) \rightarrow (S, id_S)\) is smooth (resp. unramified, resp. étale).

9. Locally affine \( S \)-spaces and schemes.

9.1. Generalities on glueing. Fix a functor \( \mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B} \).
We assume that the category \( \mathfrak{B} \) is endowed with a quasi-pretopology, \( \tau \). The latter is a function which assigns to each object \( X \) of \( \mathfrak{B} \) a family, \( \tau_X \), of covers of \( X \). An element of \( \tau_X \) is set of arrows \( \{ U_i \xrightarrow{u_i} X | i \in J \} \). We assume that any isomorphism forms a cover, and the composition of covers is a cover.

A cover of the form \( \{ \mathfrak{f}(U_i) \xrightarrow{u_i} X | i \in J \} \) of an object \( X \) is called a \((\mathfrak{A}, \tau)\)-cover, or simply \( \mathfrak{A} \)-cover, if \( \tau \) is fixed.

An object \( X \) of \( \mathfrak{B} \) is called locally \((\mathfrak{A}, \tau)\)-affine (or locally \( \mathfrak{A} \)-affine, if no ambiguity arises) if it has an \((\mathfrak{A}, \tau)\)-cover.

We denote by \( \text{Sp}_{\mathfrak{A}, \tau} \) the full subcategory of the category \( \mathfrak{B} \) whose objects are locally \((\mathfrak{A}, \tau)\)-affine.

**9.1.1. Quasi-finite locally \( \mathfrak{A} \)-affine objects.** Given a quasi-pretopology \( \tau \) on \( \mathfrak{B} \), let \( \tau_i \) denote the quasi-pretopology formed by all finite covers of \( \tau \). We call an object \( X \) of \( \mathfrak{B} \) quasi-finite locally \((\mathfrak{A}, \tau)\)-affine if it is locally \((\mathfrak{A}, \tau_i)\)-affine.

**9.1.2. 2-\( \mathfrak{A} \)-covers and 2-locally \( \mathfrak{A} \)-affine objects.** An \((\mathfrak{A}, \tau)\)-cover \( \{ \mathfrak{f}(U_i) \xrightarrow{u_i} X | i \in J \} \) will be called a 2-(\( \mathfrak{A}, \tau)\)-cover (or 2-\( \mathfrak{A} \)-cover), if for any \( i, j \in J \), there exists a set of morphisms \( (U_i, u_i) \leftarrow (U_{ij}, u_{ij}) \rightarrow (U_j, u_j) \), \( \nu \in J_{ij} \) in the category \( \mathfrak{f}/X \) such that the corresponding set of morphisms \( \{ \mathfrak{f}(U_{ij}) \rightarrow \mathfrak{f}(U_i) \times_X \mathfrak{f}(U_j) | \nu \in J_{ij} \} \) is a cover for any \( i, j \in J \). We call the diagram

\[
(U_i, u_i) \leftarrow (U_{ij}, u_{ij}) \rightarrow (U_j, u_j), \quad \nu \in J_{ij}, \quad i, j \in J,
\]

(in the category \( \mathfrak{f}/X \)) a diagram of relations of the 2-cover \( \mathfrak{A} = \{ \mathfrak{f}(U_i) \xrightarrow{u_i} X | i \in J \} \).

We call an object \( X \) of \( \mathfrak{B} \) 2-locally \( \mathfrak{A} \)-affine if it has a 2-locally \( \mathfrak{A} \)-affine cover.

**9.1.3. Weakly semiseparated covers.** We call an \( \mathfrak{A} \)-cover \( \{ \mathfrak{f}(U_i) \xrightarrow{u_i} X | i \in J \} \) weakly semiseparated if for any \( i, j \in J \), there exists a diagram

\[
(U_i, u_i) \leftarrow (U_{ij}, u_{ij}) \rightarrow (U_j, u_j)
\]

in \( \mathfrak{f}/X \) such that the square

\[
\begin{array}{ccc}
\mathfrak{f}(U_{ij}) & \longrightarrow & \mathfrak{f}(U_j) \\
\downarrow & & \downarrow \\
\mathfrak{f}(U_i) & \longrightarrow & X
\end{array}
\]

is cartesian; in particular, the object \( \mathfrak{f}(U_i) \times_X \mathfrak{f}(U_j) \) is isomorphic to an object of the form \( \mathfrak{f}(U_{ij}) \). It follows that any weakly semiseparated \( \mathfrak{A} \)-cover is a 2-\( \mathfrak{A} \)-cover.

We say that an object \( X \) of \( \mathfrak{B} \) is \( \mathfrak{A} \)-weakly semiseparated if it has a weakly semisepa-

ated \( \mathfrak{A} \)-cover.

**9.2. \( \mathfrak{A} \)-Representable morphisms and covers, and locally \( \mathfrak{A} \)-representable objects.** If \( \mathcal{E} \) is a subcategory of \( \mathfrak{B} \) such that \( \text{Ob}\mathcal{E} = \text{Ob}\mathfrak{B} \), we denote by \( \tau^\mathcal{E} \) the quasi-pretopology on \( \mathfrak{B} \) formed by all covers \( \{ U_i \xrightarrow{u_i} X | i \in J \} \) in \( \tau \) such that all morphisms \( u_i \) belong to \( \mathcal{E} \). Given a functor \( \mathfrak{A} \xrightarrow{\mathfrak{f}} \mathfrak{B} \), we have a natural choice of the subcategory \( \mathcal{E} \), which is the subcategory of \( \mathfrak{A} \)-representable morphisms described below.
We call a morphism $X \xrightarrow{f} Y$ of the category $\mathcal{B}$ $\mathfrak{A}$-representable if for any morphism $\mathfrak{F}(V) \xrightarrow{g} Y$, there exist morphisms $\mathfrak{F}(W) \xrightarrow{\tilde{g}} X$ and $W \xrightarrow{v} V$ such that

\[
\begin{array}{ccc}
\mathfrak{F}(W) & \xrightarrow{\tilde{g}} & X \\
\downarrow \mathfrak{F}(v) & & \downarrow f \\
\mathfrak{F}(V) & \xrightarrow{g} & Y \\
\end{array}
\]

is a cartesian square; in particular, it commutes.

**9.2.1. Lemma.** (a) Every isomorphism in $\mathcal{B}$ is $\mathfrak{A}$-representable.
(b) The composition of $\mathfrak{A}$-representable morphisms is $\mathfrak{A}$-representable.
(c) If in a cartesian square

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\downarrow f & & \downarrow f \\
Y' & \xrightarrow{g} & Y \\
\end{array}
\]

the morphism $X \xrightarrow{f} Y$ is $\mathfrak{A}$-representable, then the morphism $X' \xrightarrow{f'} Y'$ is $\mathfrak{A}$-representable.

**Proof.** The assertion (a) is obvious.

The assertions (b) and (c) follow from the general nonsense fact that the composition of cartesian squares is a cartesian square: if in the commutative diagram

\[
\begin{array}{ccc}
X'' & \xrightarrow{f''} & X' & \xrightarrow{f'} & X \\
\downarrow \mathfrak{F}(v) & & \downarrow f' & & \downarrow f \\
\mathfrak{F}(V) & \xrightarrow{g} & Y' & \xrightarrow{f} & Y \\
\end{array}
\]

both squares are cartesian, then the square

\[
\begin{array}{ccc}
X'' & \xrightarrow{f''} & X \\
\downarrow \mathfrak{F}(v) & & \downarrow f \\
\mathfrak{F}(V) & \xrightarrow{g} & Y \\
\end{array}
\]

is cartesian. Details are left to the reader. ■

**9.2.2. Representable covers.** We call a cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ $\mathfrak{A}$-representable if each morphism $u_i$ of the cover is $\mathfrak{A}$-representable. We denote by $\tau^\mathfrak{A}$ the function which assigns to each object $X$ of the category $\mathcal{B}$ the set, $\tau^\mathfrak{A}_X$, of $\mathfrak{A}$-representable covers of $X$.

**9.2.2.1. Lemma.** The function $\tau^\mathfrak{A}$ is a quasi-pretopology on $\mathcal{B}$. If $\tau$ is a pretopology, then $\tau^\mathfrak{A}$ is a pretopology.

**Proof.** The assertion is a corollary of 9.2.1. ■

**9.2.3. Locally $\mathfrak{A}$-representable objects.** Evidently, every representable $\mathfrak{A}$-cover is weakly semiseparated (cf. 9.1.3.). In particular, it is a 2-$\mathfrak{A}$-cover.
We say that an object $X$ of $\mathcal{B}$ is \textit{locally $\mathcal{A}$-representable} if it has a representable $\mathcal{A}$-cover. Thus, every locally $\mathcal{A}$-representable object is locally $\mathcal{A}$-affine.

9.3. Coinduced pretopology. Let $\mathcal{A} \xrightarrow{\Phi} \mathcal{B}$ be a functor, and let $\mathcal{T}$ be a quasi-pretopology on $\mathcal{A}$. The coinduced quasi-pretopology, $\Phi^*\mathcal{T}$, on $\mathcal{B}$ is defined as follows: a set of arrows $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is a cover of $X$ iff for any morphism $\Phi(V) \xrightarrow{g} X$, there exists a cover $\{V_j \xrightarrow{v_j} V \mid j \in I\} \in \mathcal{T}_V$ such that for every $j \in I$, the morphism $g \circ \Phi(v_j) : \Phi(V_j) \to X$ factors through $u_i$ for some $i \in J$ (cf. [R4, 4.4]).

9.3.1. Proposition. Suppose $\mathcal{B}$ is a category with fiber products. Then the coinduced quasi-pretopology $\Phi^*\mathcal{T}$ on $\mathcal{B}$ is a pretopology.

Proof. Let $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a cover in $\Phi^*\mathcal{T}$ and $Y \xrightarrow{g} X$ an arbitrary morphism. The claim (equivalent to the proposition) is that the set of arrows $\{U_i \times_X Y \xrightarrow{u_i} Y \mid i \in J\}$ is a cover in $\Phi^*\mathcal{T}$.

In fact, let $\Phi(V) \xrightarrow{v} Y$ be an arbitrary morphism. Since $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is a cover, there exists a cover $\{V_j \xrightarrow{v_j} V \mid j \in I\} \in \mathcal{T}$ such that for any $j \in I$, there exists $i_j \in J$ and a morphism $\Phi(V_j) \xrightarrow{\bar{v}_j} U_{i_j}$ which make the diagram

$$
\begin{array}{ccc}
\Phi(V_j) & \xrightarrow{\bar{v}_j} & U_{i_j} \\
\Phi(v_j) \downarrow & & \downarrow u_{i_j} \\
\Phi(V) & \xrightarrow{g \circ v} & X
\end{array}
$$

commute. The commutativity of (1) implies the existence of a unique morphism $\Phi(V_j) \xrightarrow{v'} U_{i_j} \times_X Y$ such that the diagram

$$
\begin{array}{ccc}
\Phi(V_j) & \xrightarrow{\bar{v}_j'} & U_{i_j} \times_X Y \\
\Phi(v_j) \downarrow & & \downarrow u_{i_j} \\
\Phi(V) & \xrightarrow{v} & Y
\end{array}
$$

commutes and the morphism $\Phi(V_j) \xrightarrow{\bar{v}_j} U_{i_j}$ is the composition of $\Phi(V_j) \xrightarrow{v'} U_{i_j} \times_X Y$ and the canonical projection $U_{i_j} \times_X Y \to U_{i_j}$. This shows that $\{U_i \times_X Y \xrightarrow{u_i} Y \mid i \in J\}$ is a cover in $\Phi^*\mathcal{T}$. $\blacksquare$

9.3.2. Note. Let $\mathcal{A}$ be a category with fiber products and $\mathcal{T}$ a quasi-pretopology on $\mathcal{A}$. Taking $\Phi=Id_{\mathcal{A}}$, we obtain the coinduced pretopology, $\mathcal{T}^g$, on $\mathcal{B}=\mathcal{A}$. The pretopology $\mathcal{T}^g$ is the finest pretopology among those pretopologies on $\mathcal{A}$ which are coarser than $\mathcal{T}$.


9.4.1. Geometric spaces and schemes. Let $\mathcal{B}$ be the category of locally ringed topological spaces which we call otherwise \textit{geometric spaces}, $\mathcal{A}$ the category opposite to

51
the category $CRings$ of commutative unital rings, $\mathfrak{A}$ the functor $\mathfrak{A} \to \mathfrak{B}$ which assigns to every commutative ring its spectrum. The pretopology on $\mathfrak{B}$ is the standard Zariski pretopology given by families of open immersions covering the underlying space: a set $\{(U_i, \mathcal{O}_{U_i}) \to (X, \mathcal{O}_X) \mid i \in J\}$ of open immersions is a cover iff $\bigcup_{i \in J} U_i = X$.

Then locally $\mathfrak{A}$-affine objects of $\mathfrak{B}$ are arbitrary schemes.

### 9.4.1.1. Semiseparated schemes.** Locally $\mathfrak{A}$-representable objects of $\mathfrak{B}$ are precisely semiseparated schemes. Recall that a scheme $X = (X, \mathcal{O})$ is called semiseparated if it has an affine cover $\{U_i \to X \mid i \in J\}$ such that each morphism $U_i \to X$ is representable. Clearly, every semiseparated scheme is weakly separated.

### 9.4.2. Quasi-finite $\mathfrak{A}$-objects. Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{A} \to \mathfrak{B}$ be as in 9.4.1. Then quasi-finite $\mathfrak{A}$-objects (i.e. locally $(\mathfrak{A}, \tau_3)$)-affine objects, where $\tau_3$ is the subpretopology of $\tau_3$ formed by finite covers, cf. 9.1.1) are exactly quasi-compact schemes.

Notice that 2-locally $(\mathfrak{A}, \tau_3)$-affine objects are quasi-compact quasi-separated schemes.

### 9.4.3. Spaces as sheaves of sets. Let $\mathfrak{A}$ be the category $CRings^{op}$, as in 9.2.1. Let $\mathfrak{B}$ be category $Esp$ of sheaves of sets on $CRings^{op}$ for the fpqc topology, and let $\mathfrak{A} \to \mathfrak{B}$ be the Yoneda embedding: $R \to CRings(R, -)$ (see 8.4.1).

Zariski covers in $CRings^{op} = \mathfrak{A}$ are given by sets of morphisms $\{R \to R_i \mid i \in J\}$ such that $R_i$ is a localization of $R$ at an element of $R$ (that is at the multiplicative set generated by this element), and $\bigcup_{i \in J} \text{Spec}(R_i) = \text{Spec}(R)$. Zariski covers form a (Zariski) pretopology, $\mathfrak{T}_3$. We define Zariski pretopology on $\mathfrak{B} = Esp$ as the pretopology coinduced by $\mathfrak{T}_3$ (cf. 9.3).

Locally affine $\mathfrak{A}$-objects in this setting are schemes in the sense of [DG], that is schemes realized as functors $CRings \to Sets$. The functor $\mathcal{S}$ which assigns to each geometric space $X = (X, \mathcal{O})$ the functor $R \mapsto \text{Hom}((\text{Spec}R, \mathcal{O}_R), X)$ establishes an equivalence between geometric schemes and functorial schemes.

Representable morphisms in $\mathfrak{B}$ are corepresentable functors $CRings \to Sets$. The functor $\mathcal{S}$ induces an equivalence of the category of semiseparated schemes (cf. 9.4.1.1) and the category of locally $\mathfrak{A}$-representable objects of $\mathfrak{B}$.

Replacing the Zariski pretopology $\mathfrak{T}_3$ by its finite version, $\mathfrak{T}_{3,\text{fpqc}}$, we obtain a full subcategory of the category of locally $\mathfrak{A}$-affine objects formed by quasi-finite locally $\mathfrak{A}$-affine objects. The functor $\mathcal{S}$ induces an equivalence of this category and the category of quasi-compact geometric schemes. The functor $\mathcal{S}$ induces an equivalence of the category of 2-locally $(\mathfrak{A}, \mathfrak{T}_{3,\text{fpqc}})$-affine objects and the category of quasi-compact, quasi-separated geometric schemes.

### 9.5. Standard noncommutative examples.** We take as $\mathfrak{A}$ the category $\mathfrak{Aff}_k = \text{Alg}_k^{op}$ opposite to the category of associative unital $k$-algebras, together with one of the canonical quasi-pretopologies defined below.

### 9.5.1. Canonical quasi-pretopologies on $\mathfrak{Aff}_k = \text{Alg}_k^{op}$. We call the image in $\mathfrak{Aff}_k$ of a set of $k$-algebra morphisms $\{R \to R_i \mid i \in J\}$ an fpqc cover if all morphisms $R \to R_i$ are flat (i.e. $R_i$ is a flat right $R$-module), and there is a finite subset $I$ of $J$
such that the family of functors \( \{ R_i \otimes_R \mid i \in I \} \) is conservative. The composition of \( \text{fpqc} \) covers is an \( \text{fpqc} \) cover, and any faithfully flat ring morphism (in particular, any ring isomorphism) \( R \rightarrow S \) forms an \( \text{fpqc} \) cover. Thus, \( \text{fpqc} \) covers form a quasi-pretopology which we denote by \( \tau_{\text{fpqc}} \).

We call an \( \text{fpqc} \) cover \( \{ \text{Spec} R_i \rightarrow \text{Spec} R \mid i \in J \} \) an \( \text{lqc} \) cover if the corresponding ring morphisms \( R \rightarrow R_i \) are localizations, or, equivalently, the corresponding ‘restriction of scalars’ functor \( R_i - \text{mod} \rightarrow R - \text{mod} \) is full (hence fully faithful). It follows from [R4, 2.6.3.1] that \( \text{lqc} \) covers form pretopology on \( \text{Aff} \) which we denote by \( \tau_{\text{lqc}} \).

We call an \( \text{fpqc} \) cover \( \{ R \rightarrow R_i \mid i \in J \} \) an \( \text{fppf} \) cover if it consists of finitely presentable morphisms. We denote the \( \text{fppf} \) quasi-pretopology by \( \tau_{\text{fppf}} \).

A set of algebra morphisms \( \{ R \rightarrow R_i \mid i \in J \} \) defines a Zariski cover if it consists of finitely presentable localizations and the family of functors \( \{ R_i \otimes_R \mid i \in J \} \) is conservative. Zariski covers form a pretopology which we denote by \( \tau_3 \) and call it the Zariski pretopology.

9.5.2. Noncommutative schemes as presheaves of sets. Let \( \mathcal{B} \) be the category \( \mathcal{E} \mathcal{s}p_{\mathcal{N}C} \) of sheaves of sets on \( \text{Aff}_k \) for \( \text{fpqc} \) quasi-pretopology. In other words, objects of \( \mathcal{B} \) are functors \( \text{Rings} \rightarrow \text{Sets} \) which preserve finite products, and for any faithfully flat ring morphism \( R \rightarrow T \), the diagram

\[
X(R) \longrightarrow X(T) \longrightarrow X(T \star_R T)
\]

is exact. The functor \( \mathfrak{A} \xrightarrow{\mathfrak{F}} \mathcal{B} \) is the Yoneda embedding, \( R \mapsto \text{Alg}_k(R, -) \) (see 8.4.2).

Let \( \tau_3^\circ \) denote the pretopology on \( \mathcal{B} \) coinduced by the Zariski pretopology \( \tau_3 \) via the functor \( \mathfrak{F} \). We define schemes as locally \( (\mathfrak{A}, \tau_3^\circ) \)-affine objects of \( \mathcal{B} \).

9.5.3. Remark on \( \text{fpqc} \)-locally affine spaces. Let the category \( \mathcal{B} \) and the functor \( \mathfrak{A} \xrightarrow{\mathfrak{F}} \mathcal{B} \) be the same as in 9.5.2. But, we take the \( \text{fpqc} \) quasi-pretopology on \( \mathfrak{A} = \text{Aff}_k \) instead of the Zariski pretopology. Let \( \tau_{\text{fpqc}}^\circ \) denote the pretopology coinduced on \( \mathcal{B} \) by the \( \text{fpqc} \) quasi-pretopology on \( \mathfrak{A} = \text{Aff}_k \). Applying the formalism of 9.1, we obtain locally \( (\mathfrak{A}, \tau_{\text{fpqc}}^\circ) \)-affine spaces.

This approach, however, is less satisfactory in the case of general \( \text{fpqc} \) covers, than in the case of Zariski covers. The reason is that \( \text{fpqc} \) covers do not form a pretopology; hence the operation of coinduction decimates the original quasi-pretopology on \( \text{Aff}_k \). Fortunately, this inconvenience is easily avoided by defining \( \text{fpqc} \) quasi-pretopology directly on the category \( \mathcal{B} \).

9.6. Flat quasi-pretopologies in \( |\text{Cat}|^\circ \). Let \( \mathcal{B} = |\text{Cat}|^\circ \). We call a set of morphisms \( \{ U_i \xrightarrow{u_i} X \mid i \in J \} \) in \( |\text{Cat}|^\circ \) a weakly flat cover if all \( u_i \) are weakly flat and the set of their inverse image functors, \( \{ u_i^* \mid i \in J \} \), is conservative. This defines a weakly flat quasi-pretopology, \( \tau^w \), on the category \( |\text{Cat}|^\circ \).

9.6.1. Finiteness conditions. We call a weakly flat cover an \( \text{fpqc} \) cover, if it contains a finite subcover. We denote the corresponding quasi-pretopology by \( \tau_{\text{fpqc}} \).

Let \( \mathcal{E} \) be a set of types of diagrams. We denote by \( \tau_{\text{fpqc}}^\mathcal{E} \) the quasi-pretopology defined as follows: \( \{ U_i \xrightarrow{u_i} X \mid i \in J \} \) belongs to \( \tau_{\text{fpqc}}^\mathcal{E} \) iff it is a weakly flat \( \text{fpqc} \) cover such that all direct image functors, \( u_i^* \), preserve colimits from \( \mathcal{E} \).
We denote by $\tau_{\text{fpqc}}$ the quasi-pretopology formed by weakly flat fpqc covers which consist of affine morphisms. We denote by $\tau^u$ the quasi-pretopology generated by weakly flat covers which consist of one morphism.

Finally, we denote by $\tau_{\text{af}}$ the quasi-pretopology generated by weakly flat covers which consist of one affine morphism.

### 9.6.2. Semiseparated covers

We call a weakly flat cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ semiseparated if all morphisms $u_i$ are affine. We denote the corresponding quasi-pretopology by $\tau_{\text{af}}$.

### 9.6.3. Proposition

Let $\mathbb{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a weakly flat cover and $\mathcal{U} = \prod_{i \in J} U_i \xrightarrow{u} X$ the canonical morphism corresponding to the cover $\mathbb{U}$.

(a) If the category $C_X$ has products of $J$ objects, then the morphism $\mathcal{U} \xrightarrow{u} X$ is weakly flat and conservative.

(b) If the category $C_X$ is additive and the cover $\mathbb{U}$ is finite and semiseparated (i.e. every morphism $u_i$ is affine), then the morphism $\mathcal{U} \xrightarrow{u} X$ is affine.

**Proof.** The family of inverse image functors $\mathbb{U} = \{C_X \xrightarrow{u_i} C_{U_i} \mid i \in J\}$ is conservative iff the corresponding functor

$$C_X \xrightarrow{u^*} \prod_{i \in J} C_{U_i} = C_{\mathcal{U}}, \quad x \mapsto (u_i^*(x)|i \in J),$$

is conservative. Similarly, the functors $u_i^*$ preserve kernels of coreflexive pairs of arrows for all $i \in J$ iff the functor $u^*$ has the same property.

(a) Suppose the category $C_X$ has products of $J$ objects. Then the functor $u_* : (a_i|i \in J) \mapsto \prod_{i \in J} u_i^*(a_i)$ is a right adjoint to the functor $u^*$.

(b) If every direct image functor $u_i^*$ is conservative, then the functor $u_*$ is conservative. If the category $C_X$ is additive and the cover $\mathbb{U}$ is finite (i.e. $J$ is finite), then $u_* (a_i|i \in J) = \prod_{i \in J} u_i^*(a_i)$ for any object $(a_i|i \in J)$ of the category $C_{\mathcal{U}}$, and for any $x \in Ob C_X$, we have:

$$C_X(u_*(a_i|i \in J), x) = C_X(\prod_{i \in J} u_i^*(a_i), x) \simeq \prod_{i \in J} C_X(u_i^*(a_i), x) \simeq \prod_{i \in J} C_X(a_i, u_i^1(x)) = C_{\mathcal{U}}((a_i|i \in J), (u_i^1(x)|i \in J)).$$

Here $u_i^1$ is a right adjoint to the direct image functor $u_i^*$. This shows that the functor

$$C_X \xrightarrow{u'} C_{\mathcal{U}}, \quad x \mapsto (u_i^1(x)|i \in J),$$

is a right adjoint to the functor $u_*$. □

### 9.7. Locally affine morphisms. Relative schemes

Fix a 'space' $S$, and consider the category $\mathfrak{B} = |\text{Cat}|_S$. Recall that $|\text{Cat}|_S$ is a full subcategory of $|\text{Cat}|_o/S$ whose objects are pairs $(X, f)$, where $X \xrightarrow{f} S$ is a continuous morphism.
We assume that the category $C_S$ has cokernels of reflexive pairs of arrows. There are two extreme choices of the category of ‘local’ (or ‘affine’) objects $\mathfrak{A}$. The largest (in a certain sense) choice is the category $\mathfrak{A} = \text{Aff}_S'$ of weakly affine morphisms to $S$. The other extremity is the category $\text{Aff}_S$ of affine morphisms to $S$ (as in 8.5.3). Intermediate choices are categories $\text{Aff}_S^\mathfrak{S}$, where $\mathfrak{S}$ is a set of types of diagrams: objects of $\text{Aff}_S^\mathfrak{S}$ are pairs $(X, f)$, where $f$ is a continuous morphism $X \rightarrow S$ such that $f_*$ is weakly affine and preserves colimits of diagrams of $\mathfrak{S}$.

In each of these cases, the functor $\mathfrak{A} \xrightarrow{\mathfrak{S}} \mathfrak{B}$ is the inclusion functor.

Any quasi-pretopology on $|\text{Cat}|^\mathfrak{o}$ induces a quasi-pretopology on $|\text{Cat}|^\mathfrak{o}/S$. In particular, each of the canonical quasi-pretopologies on $|\text{Cat}|^\mathfrak{o}$ defined in 9.7 induces a canonical quasi-pretopology on $\mathfrak{B} = |\text{Cat}|^\mathfrak{o}_S$. Thus, we have the quasi-pretopology $\tau_{\text{fpqc}}^\mathfrak{A}$ given by finite weakly flat covers and its versions, $\tau_{\text{fpqc}}^\mathfrak{S}$ and $\tau_{\text{fpqc}}^\mathfrak{S}$ (cf. 9.6.1).

**9.7.1. Locally affine $S$-‘spaces’.** We call an object $(X, X \xrightarrow{f} S)$ of the category $\mathfrak{B} = |\text{Cat}|^\mathfrak{o}_S$ a locally affine $S$-‘space’, if it is a locally $\mathfrak{A}$-affine object of $|\text{Cat}|^\mathfrak{o}_S$ for the quasi-pretopology $\tau_{\text{fpqc}}$, and $\mathfrak{A} = \text{Aff}_S$. An object $(X, X \xrightarrow{f} S)$ of $|\text{Cat}|^\mathfrak{o}_S$ is called a semiseparaed locally affine $S$-‘space’, if it is a locally $\mathfrak{A}$-affine object of $|\text{Cat}|^\mathfrak{o}_S$ with respect to the quasi-pretopology $\tau_{\text{fpqc}}^{\mathfrak{af}}$ and $\mathfrak{A} = \text{Aff}_S$. In other words, $(X, X \xrightarrow{f} S)$ has a finite weakly flat $\text{Aff}_S$-affine cover which consists of affine morphisms.

Restricting to $\text{fpqc}$ covers $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ such that all morphisms $u_i$ are flat localizations, we obtain quasi-pretopologies $\tau_{\text{fpqc}}, \tau_{\text{fpqc}}^\mathfrak{S}$, and $\tau_{\text{fpqc}}^{\mathfrak{af}}$.

**9.7.2. Relative schemes.** We call a conservative set $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ of morphisms of $|\text{Cat}|^\mathfrak{o}_S$ a Zariski cover, if all morphisms $u_i$ are locally finitely presentable localizations. Zariski covers form a quasi-pretopology, $\tau_3$.

We call an object $(X, X \xrightarrow{f} S)$ of the category $|\text{Cat}|^\mathfrak{o}_S$ an $S$-scheme, if it is a locally $\mathfrak{A}$-affine object of $|\text{Cat}|^\mathfrak{o}_S$ with respect to the quasi-pretopology $\tau_3$ and $\mathfrak{A} = \text{Aff}_S$. Taking Zariski covers which are $\text{fpqc}$ covers, we obtain the corresponding versions of Zariski quasi-pretopology: $\tau_3, \tau_{\text{fpqc}}^\mathfrak{S},$ and $\tau_{\text{fpqc}}^{\mathfrak{af}}$.

We call an object $(X, X \xrightarrow{f} S)$ of $|\text{Cat}|^\mathfrak{o}/S$ a semiseparaed $S$-scheme, if it is a locally $\mathfrak{A}$-affine object of $|\text{Cat}|^\mathfrak{o}/S$ with respect to the quasi-pretopology $\tau_3^{\mathfrak{af}}$ and $\mathfrak{A} = \text{Aff}_S$. In other words, $(X, X \xrightarrow{f} S)$ has a finite weakly flat $\text{Aff}_S$-affine Zariski cover which consists of affine localizations.

**9.8. Locally $S$-affine ‘spaces’ and schemes.** Fix an object $S$ of the category $\mathfrak{B} = |\text{Cat}|^\mathfrak{o}$. One can take as $\mathfrak{A}$ the category $\text{Aff}_S'$, or $\text{Aff}_S$, or $\text{Aff}_S^\mathfrak{S}$ for some set $\mathfrak{S}$ of types of diagrams. This time, $\mathfrak{A} \xrightarrow{\mathfrak{S}} \mathfrak{B}$ is the natural functor which maps an object $(X, X \rightarrow S)$ of the category $\mathfrak{A}$ to $X$. We call an object $X$ of $\mathfrak{B}$ $S$-locally affine ‘space’ if it is a locally $(\mathfrak{A}, \mathfrak{S})$-affine object with respect to the quasi-pretopology $\tau_{\text{fpqc}}$.

We call a conservative set $\{\mathfrak{S}(U_i) \xrightarrow{u_i} X \mid i \in J\}$ of morphisms of $|\text{Cat}|^\mathfrak{o}_S$ a Zariski cover, if all morphisms $u_i$ are $(\mathfrak{A}, \mathfrak{S})$-finitely presentable localizations. Zariski covers form a quasi-pretopology which we denote by $\tau_3$, as in the relative case sketched in 9.7.2.
We call an object \((X, X \xrightarrow{f} S)\) of the category \(|\text{Cat}|^o_S\) an \(S\)-scheme, if it is a locally \(A\)-affine object of \(|\text{Cat}|^o_S\) with respect to the quasi-pretopology \(\tau_3\) and \(A = \text{Aff}_S\).

### 9.8.1. Locally affine \(Z\)-spaces and \(Z\)-schemes.

Let \(S = \text{Sp}Z\) (i.e. \(C_S\) is the category of abelian groups). We call an object \(X\) of \(\mathfrak{B}\) locally affine \(Z\)-space (resp. a \(Z\)-scheme) if it is a locally \(A\)-affine object with respect to the quasi-pretopology \(\tau_{fpqc}\) (resp. the Zariski quasi-pretopology \(\tau_3\)).

### 9.8.2. Note.

Let \(\mathfrak{A}\) be the category \(\text{Rings}^{op}\), and let \(\mathfrak{F}\) be the functor \(\text{Sp}\) which assigns to each associative unital ring its categoric spectrum. Then the category of \(\mathfrak{A}\)-affine \(\mathfrak{F}\)-comodules.

Let \(\mathfrak{A}\) be the category \(\text{Rings}^{op}\), and let \(\mathfrak{F}\) be the functor \(\text{Sp}\) which assigns to each associative unital ring its categoric spectrum. Then the category of \(\mathfrak{A}\)-affine \(\mathfrak{F}\)-comodules.

### 9.9. The structure of locally affine \(\mathfrak{A}\)-spaces.

Fix a \(\mathfrak{A}\)-space \(S\). Let \(\mathfrak{A}\) be the category \(\text{Aff}_S\) of affine morphisms to \(S\), or the category \(\text{Aff}_{S\mathfrak{F}}\) for some set of diagram types \(\mathfrak{G}\), and let \(\mathfrak{F} \xrightarrow{\varphi} \mathfrak{B} = |\text{Cat}|^o\) be the forgerful functor \((X, X \rightarrow S) \mapsto X\).

Suppose that \(C_S\) has finite products and cokernels of reflexive pairs of arrows. Let \(\{\varphi(U_i, \bar{u}_i) = u_i \overset{\sim}{\rightarrow} X \mid i \in J\}\) be a finite \(\mathfrak{A}\)-cover of \(X\), and let \(U = \bigsqcup_{i \in J} U_i \overset{u}{\rightarrow} X\) be the corresponding morphism.

If \(C_X\) has finite products, then \(u\) is a continuous morphism (see 9.6.3), hence \(u\) is weakly flat and conservative. By Beck’s theorem, \(X\) is isomorphic to \(\text{Sp}^o(U\setminus \mathcal{G}_u)\); i.e. the category \(C_X\) is equivalent to the category of \((U\setminus \mathcal{G}_u)\)-comodules.

### 10. \(\mathfrak{A}\)-spaces of morphisms and flat descent.

#### 10.1. \(\mathfrak{A}\)-spaces of morphisms.

For any two objects \(X, Y\) of the category \(|\text{Cat}|^o\), we denote by \(\text{Hom}(X, Y)\) the object of \(|\text{Cat}|^o\) such that \(C_{\text{Hom}(X, Y)}\) is the category \(\text{Cat}^{op}(X, Y)\) of (inverse image) functors from \(C_Y\) to \(C_X\).

Let \(\bullet\) be the standard initial object of the category \(|\text{Cat}|^o\), i.e. \(C_{\bullet}\) is the category with one morphism. For any object \(X\) of \(|\text{Cat}|^o\), there are natural isomorphisms:

\[
\text{Hom}(\bullet, X) \simeq \bullet \quad \text{and} \quad \text{Hom}(X, \bullet) \simeq X.
\]

As it is observed in 2.2, \(|\text{Cat}|^o(X, \bullet) \simeq |X|\), where \(|X|\) is the set of isomorphism classes of objects of \(C_X\). In particular, \(|\text{Cat}|^o(\text{Hom}(X, Y), \bullet) \simeq |\text{Cat}|^o(X, Y)\).

Let \(X, Y, Z\) be objects of \(|\text{Cat}|^o\). The composition of functors (1-morphisms of the category \(\text{Cat}\))

\[
\text{Cat}(C_X, C_Y) \times \text{Cat}(C_Y, C_Z) \longrightarrow \text{Cat}(C_X, C_Z)
\]

can be rewritten as

\[
C_{\text{Hom}(Y, X)} \times C_{\text{Hom}(Y, X)} = C_{\text{Hom}(Y, X)} \bigsqcup_{\text{Hom}(Z, Y)} \longrightarrow C_{\text{Hom}(Z, X)}
\]

and interpreted as a inverse image functor of a morphism

\[
\text{Hom}(Z, X) \longrightarrow \text{Hom}(Y, Z) \bigsqcup \text{Hom}(X, Y)
\] (1)
Applying to (1) the functor \( |\mathsf{Cat}|^o(-, \bullet) \), we obtain the composition in \( |\mathsf{Cat}|^o \):
\[
|\mathsf{Cat}|^o(Y, X) \times |\mathsf{Cat}|^o(Z, Y) \longrightarrow |\mathsf{Cat}|^o(Z, X).
\]

10.1.1. Remark. One can view \( \mathsf{Cat} \) as a monoidal category with the monoidal structure given by the product of categories. The unit object is the category \( C_\bullet \), and for any two objects, \( C_X, C_Y \), of the category \( \mathsf{Cat} \), the category \( \mathsf{Hom}(X, Y) = \mathsf{Hom}(C_Y, C_X) \) of functors \( C_Y \rightarrow C_X \) is the inner hom from \( C_Y \) to \( C_X \). Thus, we have an enriched monoidal category. Formally dualizing this structure, we obtain the one described above, i.e. the category \( |\mathsf{Cat}|^o \) enriched by 'spaces' of morphisms \( \mathsf{Hom}(X, Y) \) for any pair of objects \( X, Y \).

10.1.2. Morphisms of \( S \)-'spaces'. Fix an object \( S \) of the category \( |\mathsf{Cat}|^o \) and consider the category \( |\mathsf{Cat}|^o/S \) of objects over \( S \). For any two objects, \( X = (X, f), Y = (Y, g) \) of \( |\mathsf{Cat}|^o/S \), we denote by \( \mathsf{Hom}_S(X, Y) \) the object of \( |\mathsf{Cat}|^o \) such that \( \mathsf{Hom}_S(X, Y) \) is the full subcategory of the category \( \mathsf{C}_\mathsf{Hom}(X, Y) = \mathsf{Hom}(C_Y, C_X) \) formed by those functors \( \phi^*: C_Y \rightarrow C_X \) for which the diagram
\[
\begin{array}{ccc}
C_Y & \xrightarrow{\phi^*} & C_X \\
g^* \downarrow & & \nearrow f^* \\
C_S & & \\
\end{array}
\]
quasi commutes. If \( \phi^* \) is an object of \( \mathsf{C}_\mathsf{Hom}_S(X, Y) \) and \( \psi^* \) is an object of \( \mathsf{C}_\mathsf{Hom}_S(Y, Z) \), then \( \phi^*\psi^* \) is an object of \( \mathsf{C}_\mathsf{Hom}_S(X, Z) \). Thus, we have the enriched category \( |\mathsf{Cat}|^o/S \) of \( S \)-'spaces' such that the forgetful functor \( |\mathsf{Cat}|^o/S \longrightarrow |\mathsf{Cat}|^o \) defines a morphism of enriched categories.

10.2. Certain enriched categories of morphisms. For any two objects \( X \) and \( Y \) of the category \( |\mathsf{Cat}|^o \), we denote by \( \mathsf{Hom}^w(X, Y), \mathsf{Hom}_w(X, Y), \) and \( \mathsf{Hom}^c(X, Y) \) objects of \( |\mathsf{Cat}|^o \) defined as follows:
\( \mathsf{C}_\mathsf{Hom}^w(X, Y) \) is the full subcategory of \( \mathsf{C}_\mathsf{Hom}(X, Y) \) formed by weakly right exact functors \( C_Y \rightarrow C_X \), i.e. functors which preserve cokernels of reflexive pairs of arrows;
\( \mathsf{C}_\mathsf{Hom}_w(X, Y) \) is the full subcategory of \( \mathsf{C}_\mathsf{Hom}(X, Y) \) formed by weakly left exact functors, i.e. functors \( C_Y \rightarrow C_X \) which preserve kernels of coreflexive pairs of arrows;
\( \mathsf{C}_\mathsf{Hom}^c(X, Y) \) is the full subcategory of \( \mathsf{C}_\mathsf{Hom}(X, Y) \) whose objects are continuous functors \( C_Y \rightarrow C_X \). In particular, \( \mathsf{C}_\mathsf{Hom}^c(X, Y) \) is a full subcategory of the category \( \mathsf{C}_\mathsf{Hom}^w(X, Y) \).

Continuous morphisms play a special role, as the following example shows.

10.2.1. Example. For any unital associative rings \( R, S \), there is a natural isomorphism \( \mathsf{Hom}^c(\mathsf{Sp}(R), \mathsf{Sp}(S)) \cong \mathsf{Sp}(S \otimes R^o) \).

In fact, the category of continuous functors from \( S - \mathsf{mod} = \mathsf{C}_\mathsf{Sp}(S) \) to \( R - \mathsf{mod} = \mathsf{C}_\mathsf{Sp}(R) \) is equivalent to the category of \( (R, S) \)-bimodules, which, in turn, is isomorphic to the category of \( R \otimes S^o \)-modules.

10.2.1. Note. Suppose that \( S \) and \( R \) are commutative rings. Then \( S \otimes R^o = S \otimes R \) is a commutative ring. Thus, if \( U \) and \( V \) are spectra of commutative rings, then \( \mathsf{Hom}^c(U, V) \) is the spectrum of a commutative ring.
10.2.2. Generalizations of $\text{Hom}^w$ and $\text{Hom}_w$. Let $\mathcal{S}$ and $\mathcal{I}$ be sets of types of diagrams. For any two objects, $X$, $Y$ of $|\text{Cat}|$, we denote by $\text{Hom}_{\mathcal{S}}(X,Y)$ (resp. by $\text{Hom}_{\mathcal{I}}(X,Y)$) the object of $|\text{Cat}|$ such that $\text{C}_\mathcal{S}\text{Hom}_{\mathcal{S}}(X,Y)$ (resp. $\text{C}_\mathcal{I}\text{Hom}_{\mathcal{I}}(X,Y)$) is a full subcategory of $\text{C}_\mathcal{I}\text{Hom}_{\mathcal{I}}(X,Y)$ whose objects are all functors $C_Y \to C_X$ which preserve colimits of diagrams from $\mathcal{I}$ (resp. limits of diagrams from $\mathcal{S}$). Besides $\text{Hom}^w$ and $\text{Hom}_w$, the cases of special interest are when $\mathcal{S}$ (or $\mathcal{I}$, or both) consist of all finite diagrams, or $\mathcal{S}$ consists of countable diagrams.

We denote by $\text{Hom}_{\mathcal{S}}^{\mathcal{I}}(X,Y)$ the intersection (or, rather, cointersection) of $\text{Hom}_{\mathcal{S}}(X,Y)$ and $\text{Hom}_{\mathcal{I}}(X,Y)$ which is the object of $|\text{Cat}|$ defined by

$$C_{\text{Hom}_{\mathcal{S}}^{\mathcal{I}}(X,Y)} = C_{\text{Hom}_{\mathcal{S}}(X,Y)} \cap C_{\text{Hom}_{\mathcal{I}}(X,Y)}$$

i.e. objects of $C_{\text{Hom}_{\mathcal{S}}^{\mathcal{I}}(X,Y)}$ are functors which preserves limits from $\mathcal{S}$ and colimits from $\mathcal{I}$. In particular, objects of $C_{\text{Hom}_{\mathcal{S}}^w(X,Y)}$ are functors $C_Y \to C_X$ which are weakly right and left exact.

10.2.3. Functoriality of these $\text{Hom}$-s. The maps

$$(X,Y) \mapsto \text{Hom}^c(X,Y) \quad \text{and} \quad (X,Y) \mapsto \text{Hom}_{\mathcal{S}}^{\mathcal{I}}(X,Y),$$

in particular, the maps $(X,Y) \mapsto \text{Hom}^w(X,Y)$ and $(X,Y) \mapsto \text{Hom}_w(X,Y)$, are functorial and stable under the composition (1), hence they define enriched subcategories of the enriched category of morphisms.

10.3. Flat descent of morphisms. Any pair $U \xrightarrow{u} X$ and $V \xrightarrow{v} Y$ of morphisms such that $v$ is continuous induces a morphism

$$(u|v) : \text{Hom}(U,V) \to \text{Hom}(X,Y)$$

having an inverse image functor $(u^*|v_*)$ defined by

$$C_{\text{Hom}(X,Y)} \longrightarrow C_{\text{Hom}(U,V)}, \quad \phi^* \longmapsto u^*\phi^*v_*.$$  \hfill (2)

10.3.1. Lemma. (a) If $U \xrightarrow{u} X$ and $V \xrightarrow{v} Y$ are continuous morphisms, then $(u|v)$ is continuous.

(b) If $u^*$ and $v_*$ are weakly continuous (i.e. they preserve cokernels of reflexive pairs of arrows), then the morphism $(u|v)$ induces a morphism

$$(u|v)_w : \text{Hom}^w(U,V) \to \text{Hom}^w(X,Y).$$

(c) If the morphisms $u$, $v$ are continuous and $v_*$ has a right adjoint (e.g. $v$ is affine), then the morphism $(u|v)$ induces a morphism

$$(u|v)_c : \text{Hom}^c(U,V) \to \text{Hom}^c(X,Y)$$

which is continuous if $u_*$ has a right adjoint.
Proof. (a) If \( u \) has a direct image, \( u_* \), then the functor

\[
(u_*|v^*): C_{\text{Hom}(U,V)} \longrightarrow C_{\text{Hom}(X,Y)}, \quad \psi^* \longmapsto u_*\phi^*v^*,
\]

is a direct image functor of the morphism \((u|v)\).

In fact,

\[
(u_*|v^*) \circ (u^*|v_*) = (u_*u^*|v_*v^*) : \phi^* \longmapsto u_*u^*\phi^*v_*v^*,
\]

and

\[
(u^*|v_*) \circ (u_*|v^*) = (u^*u_*|v^*v_*): \psi^* \longmapsto u^*u_*\psi^*v_*.
\]

Adjunction arrow \( \eta_u, \eta_v \) (resp. \( \epsilon_u, \epsilon_v \)) determine adjunction arrows

\[
\eta(u|v) : \text{Id}_{C_{\text{Hom}(X,Y)}} \longrightarrow (u|v)_* \circ (u|v)^* = (u_*u^*|v_*v^*)
\]

and

\[
\epsilon(u|v) : (u|v)^*(u|v)_* = (u^*u_*|v^*v_*) \longrightarrow \text{Id}_{C_{\text{Hom}(U,V)}},
\]

(b) If the functors \( u^*, v_* \) and \( \phi^* : C_Y \longrightarrow C_X \) preserve kernels of coreflexive pairs of arrows, then their composition, \( u^*\phi^*v_* = (u|v)^*(\phi^*) \) has the same property.

(c) If each of the functors \( u^*, v_* \) and \( \phi^* \) has a right adjoint, then the functor \( (u|v)^*(\phi^*) = u^*\phi^*v_* \) has a right adjoint. Similarly, if the functors \( u_* \) and \( \psi^* \) have right adjoints, then \( (u|v)_*(\psi^*) = u_*\psi^*v_* \) has a right adjoint. ■

10.3.2. Lemma. Let \( U \xrightarrow{u} X \) be a continuous morphism with an inverse image \( u^* \) and a direct image \( u_* \). Suppose the category \( C_X \) has kernels of coreflexive pairs of arrows and the functor \( u^* \) preserves these kernels. Then the morphism \( u \) (i.e. its inverse image \( u^* \)) is conservative iff the canonical diagram

\[
\text{Id}_{C_X} \xrightarrow{\eta_u} F_u \xrightarrow{F_u\eta_u} \xrightarrow{\eta_uF_u} F_u^2
\]

is exact. Here \( F_u = u_*u^* \) and \( \eta_u \) is an adjunction morphism.

Proof. Since the pair of arrows

\[
F_u \xrightarrow{F_u\eta_u} F_u^2 \xrightarrow{\eta_uF_u} F_u^2
\]

is coreflexive, and by hypothesis, the category \( C_X \) has kernels of coreflexive arrows, the pair (5) has a kernel, \( K_\eta \), and the morphism \( \eta : \text{Id}_{C_X} \longrightarrow F_u \) is uniquely decomposed into \( \text{Id}_{C_X} \xrightarrow{\eta_u} K_\eta \xrightarrow{\kappa_u} F_u \), where \( \kappa_u \) is the canonical morphism. Notice that the diagram

\[
u^* \xrightarrow{u^*\eta_u} u^*F_u \xrightarrow{u^*F_u\eta_u} u^*F_u^2 \xrightarrow{u^*\eta_uF_u} u^*F_u^2
\]
is exact without any additional conditions on \( u \) (except for \( u \) being continuous). Since \( u^* \) preserves kernels of coreflexive pairs of arrows, the exactness of (6) is equivalent to that \( u^*(s_u) \) is an isomorphism. Thus, if \( u^* \) is conservative, then \( s_u \) is an isomorphism, i.e. the diagram (4) is exact.

Conversely, suppose the diagram (4) is exact. Let \( s : L \to M \) be a morphism such that \( u^*(s) \) is an isomorphism. Then the vertical arrows \( F_u(s) \) and \( F^2_u(s) \) in the commutative diagram

\[
\begin{array}{ccc}
L & \xrightarrow{\eta_u(L)} & F_u(L) & \xrightarrow{\eta_u F_u(L)} & F^2_u(L) \\
F_u(s) & \downarrow & \downarrow & \downarrow & \downarrow F^2_u(s) \\
M & \xrightarrow{\eta_u(M)} & F_u(M) & \xrightarrow{\eta_u F_u(M)} & F^2_u(M)
\end{array}
\]

are isomorphisms. Since the horizontal subdiagrams in (7) are exact, this implies that \( s \) is an isomorphism.

**10.3.3. Note.** We shall also refer to 10.3.2 when the dual assertion is needed:

Let \( U \xrightarrow{u} X \) be a continuous morphism with an inverse image \( u^* \) and a direct image \( u_* \). Suppose the category \( C_U \) has cokernels of reflexive pairs of arrows and the functor \( u_* \) preserves these cokernels. Then the functor \( u^* \) is conservative iff the canonical diagram

\[
\begin{array}{ccc}
G^2_u & \xrightarrow{G^2 u \epsilon u} & G_u & \xrightarrow{\epsilon u} & Id_{C_U} \\
\end{array}
\]

is exact. Here \( G_u = u^* u_* \) and \( \epsilon u \) an adjunction morphism.

**10.3.4. Proposition.** Let \( U \xrightarrow{u} X \) and \( V \xrightarrow{v} Y \) be continuous morphisms.

(a) If \( u \) is weakly coflat (i.e. \( u_* \) preserves cokernels of reflexive pairs of arrows), then the morphism

\[
(u|v) : \text{Hom}(U, V) \to \text{Hom}(X, Y)
\]

has the same property.

(b) Suppose both \( u \) and \( v \) are weakly affine (i.e. they are weakly coflat and their direct image functors are conservative); and let the category \( C_X \) have cokernels of reflexive pairs of arrows. Then the morphism

\[
(u|v)_w : \text{Hom}^w(U, V) \to \text{Hom}^w(X, Y)
\]

is weakly affine.

(c) Suppose that \( u \) and \( v \) are weakly flat and conservative, and the category \( C_U \) has limits of coreflexive pairs of arrows. Then the morphism

\[
(u|v)^w : \text{Hom}^w(U, V) \to \text{Hom}^w(X, Y)
\]

is weakly flat and conservative.
Proof. (a) The functor

\[(u|v)_* : C_{\text{Hom}(U,V)} \longrightarrow C_{\text{Hom}(X,Y)}, \quad \phi^* \mapsto u_*\phi^*v^*,\]

preserves all colimits the functor \(u_*\) preserves, because the functor \(\phi^* \mapsto \phi^*v^*\) preserves all (small) colimits and limits.

(b) Let \(G_v = v^*v_*\), and let \(\epsilon_v : G_v \longrightarrow \text{Id}_{C_V}\) be an adjunction morphism. If \(v_*\) preserves cokernels of reflexive pairs of arrows and is conservative, then, by 10.3.3, the diagram

\[\begin{array}{ccc}
G_v^2 & \xrightarrow{G_v \epsilon_v} & G_v \\
\epsilon_v G_v & \xrightarrow{\epsilon_v} & \text{Id}_{C_V}
\end{array}\]  \hspace{1cm} (9)

is exact. If \(\phi^* : C_Y \longrightarrow C_X\) is an object of \(C_{\text{Hom}^w(X,Y)}\), i.e. \(\phi^*\) preserves cokernels of reflexive pairs of arrows, the diagram \(\phi^*(9)\) is exact. If, in addition, the morphism \(u_*\) preserves cokernels of reflexive pairs of arrows and is conservative, hence the diagram

\[\begin{array}{ccc}
G_u^2 & \xrightarrow{G_u \epsilon_u} & G_u \\
\epsilon_u G_u & \xrightarrow{\epsilon_u} & \text{Id}_{C_U}
\end{array}\]

is exact. If the category \(C_X\) has a certain type of colimits, the category \(C_{\text{Hom}(X,Y)}\) and its subcategory \(C_{\text{Hom}^w(X,Y)}\) have the same type of colimits. In particular, \(C_{\text{Hom}^w(X,Y)}\) has cokernels of reflexive pairs of arrows, because \(C_X\) has them.

Thus, the morphism \((u|v)_w : \text{Hom}^w(U,V) \longrightarrow \text{Hom}^w(X,Y)\) satisfies the conditions of 9.3.3, hence its direct image functor is conservative.

(c) The assertion (c) is dual (therefore equivalent) to the assertion (b). ■

10.3.5. Corollary. Let \(U \xrightarrow{u} X\) and \(V \xrightarrow{v} Y\) be continuous morphisms.

(a) Suppose both \(u\) and \(v\) are weakly affine; and let the category \(C_X\) have cokernels of reflexive pairs of arrows. Then the morphism

\[(u|v)_w : \text{Hom}^w(U,V) \longrightarrow \text{Hom}^w(X,Y)\]

is monadic, i.e. it is isomorphic to the canonical morphism

\[\text{Sp}(\mathcal{F}_{(u|v)_w}/\text{Hom}^w(X,Y)) \longrightarrow \text{Hom}^w(X,Y).\]

Here \(\mathcal{F}_{(u|v)_w}\) is the monad on \(\text{Hom}^w(X,Y)\) associated with the (choice of an inverse and a direct image functors and adjunction arrows of the) morphism \((u|v)_w\).

(b) Suppose that \(u\) and \(v\) are weakly flat and conservative, and the category \(C_U\) has limits of coreflexive pairs of arrows. Then the morphism

\[(u|v)^w : \text{Hom}^w(U,V) \longrightarrow \text{Hom}^w(X,Y)\]

is monadic, i.e. it is isomorphic to the canonical morphism

\[\text{Sp}(\mathcal{F}_{(u|v)^w}/\text{Hom}^w(X,Y)) \longrightarrow \text{Hom}^w(X,Y).\]
is comonadic, i.e. it is isomorphic to the canonical morphism

$$\mathcal{H} \text{om}_w(U, V) \longrightarrow \text{Sp}^o(\mathcal{H} \text{om}_w(U, V) \setminus G_{(u|v)_w}).$$

Here $G_{(u|v)_w}$ is the comonad on $\mathcal{H} \text{om}_w(U, V)$ associated with the morphism $(u|v)$.

**Proof.** (a) By 10.3.4(b), the morphism $(u|v)_w$ satisfies the conditions of the Beck’s theorem 5.4.2.

(b) Similarly, by 10.3.4(c), the morphism $(u|v)^w$ satisfies the conditions of the (dual) Beck’s theorem 5.4.1.

The assertions follow from the respective versions of the Beck’s theorem. $\blacksquare$

Denote by $\mathcal{H} \text{om}^{wFL}(X, Y)$ the object (in $\text{Cat}^{op}$) of weakly flat morphisms: objects of the category of quasi-coherent modules on $\mathcal{H} \text{om}^{wFL}(X, Y)$ are functors $\phi^* : C_Y \longrightarrow C_X$ which have a right adjoint and preserve kernels of coreflexive pairs of arrows.

**10.3.6. Proposition.** Let $U \overset{u}{\longrightarrow} X$ and $V \overset{v}{\longrightarrow} Y$ be affine weakly flat morphisms. Then a direct image functor of the morphism

$$(u|v)_c : \mathcal{H} \text{om}^{c}(U, V) \longrightarrow \mathcal{H} \text{om}^{c}(X, Y)$$

is conservative and preserves colimits of small diagrams.

The morphism $(u|v)_c$ induces a weakly flat conservative morphism

$$(u|v)^{wFL}_c : \mathcal{H} \text{om}^{wFL}(U, V) \longrightarrow \mathcal{H} \text{om}^{wFL}(X, Y)$$

whose direct image functor preserves colimits of small diagrams.

**Proof.** Since $u$ and $v$ are affine, in particular $u_*$ and $v_*$ have right adjoints, it follows from 10.3.1(c) that $(u|v)_c$ is a continuous morphism. By 10.3.4, a direct image functor of the morphism $(u|v)_c$ is conservative. $\blacksquare$

**10.3.7. Corollary.** Let $U \overset{u}{\longrightarrow} X$ and $V \overset{v}{\longrightarrow} Y$ be affine weakly flat morphisms. Then

(a) The morphism

$$(u|v)_c : \mathcal{H} \text{om}^{c}(U, V) \longrightarrow \mathcal{H} \text{om}^{c}(X, Y)$$

is isomorphic to the canonical morphism

$$\text{Sp}(F_{(u|v)_c}/\mathcal{H} \text{om}^{c}(X, Y)) \longrightarrow \mathcal{H} \text{om}^{c}(X, Y),$$

where $F_{(u|v)_c} = (F_{(u|v)_c}, \mu_{(u|v)_c})$ is a monad associated with $(u|v)_c$. The functor $F_{(u|v)_c}$ preserves colimits.

(b) The morphism $(u|v)_c$ is the composition of the standard morphism

$$\mathcal{H} \text{om}^{c}(U, V) \longrightarrow \text{Sp}^o(\mathcal{H} \text{om}^{c}(U, V) \setminus G_{(u|v)_c}).$$
and the canonical morphism

\[ \text{Sp}^\circ(\mathcal{H}om^c(U, V) \setminus \mathcal{G}_{(u|v)}) \longrightarrow \mathcal{H}om^c(X, Y). \] (11)

The morphism (11) is a localization.

(c) The morphism

\[ (u|v)_{wfl} : \mathcal{H}om^{wfl}(U, V) \longrightarrow \mathcal{H}om^{wfl}(X, Y) \]

is isomorphic to the canonical morphism

\[ \mathcal{H}om^{wfl}(U, V) \longrightarrow \text{Sp}^\circ(\mathcal{H}om^{wfl}(U, V) \setminus \mathcal{G}_{(u|v)_{wfl}}), \]

where \( \mathcal{G}_{(u|v)_{wfl}} = (G_{(u|v)_{wfl}}, \delta_{(u|v)_{wfl}}) \) is a comonad associated with \((u|v)_{wfl}\). The functor \( G_{(u|v)_{wfl}} \) preserves colimits.

**Proof.** The assertion is a consequence of 10.3.6 and the Beck’s theorem 5.4.1. ■

10.3.7.1. A description of the morphisms in 10.3.7. Since the morphisms \( U \xrightarrow{u} X \) and \( V \xrightarrow{v} Y \) are affine and weakly flat, \( X \) and \( Y \) can be identified with resp. \( \text{Sp}^\circ(U \setminus \mathcal{G}_u) \) and \( \text{Sp}^\circ(V \setminus \mathcal{G}_v) \). One can define a (fully faithful) direct image functor of the morphism (11) as follows.

Let \( \mathcal{M} = (M, \zeta) \) be a \( \mathcal{G}_{(u|v)} \)-comodule; and let \( \mathcal{L} = (L, \xi_L) \) be a \( \mathcal{G}_v \)-comodule (i.e. an object of \( C_Y \)). Notice that the \( \mathcal{G}_{(u|v)} \)-comodule structure on \( M \) is the composition of a right coaction, \( \zeta^v : M \longrightarrow MG_v \), and a left coaction, \( \zeta_u : M \longrightarrow G_u M \). In other words, \( \mathcal{M} \) is naturally regarded as a \(((U \setminus \mathcal{G}_u), (V \setminus \mathcal{G}_v))\)-bicomodule \((M, \zeta_u, \zeta^v)\).

Thus, we have a coreflexive pair of morphisms

\[ M(L) \xrightarrow{\zeta^v(L)} MG_v(L). \] (12)

The comodule structure \( \zeta_u : M \longrightarrow G_u M \) induces \( \mathcal{G}_u \)-comodule structure on \( M(L) \) and \( MG_v(L) \), so that (12) becomes a diagram of \( \mathcal{G}_u \)-comodules. Since the functor \( G_u = u^*u_* \) preserves kernels of coreflexive pairs of arrows, the kernel \( \tilde{\mathcal{M}}(\mathcal{L}) \) of the (coreflexive) pair (12) has a (necessarily unique) \( \mathcal{G}_u \)-comodule structure, \( \xi_{\tilde{\mathcal{M}}(\mathcal{L})} : \tilde{\mathcal{M}}(\mathcal{L}) \longrightarrow G_u \tilde{\mathcal{M}}(\mathcal{L}) \), such that the canonical morphism \( \tilde{\mathcal{M}}(\mathcal{L}) \longrightarrow \mathcal{M}(\mathcal{L}) \) is a \( \mathcal{G}_u \)-comodule morphism.

The correspondence \( (\mathcal{M}, \mathcal{L}) \mapsto (\tilde{\mathcal{M}}(\mathcal{L}), \xi_{\tilde{\mathcal{M}}(\mathcal{L})}) \) is functorial in \( \mathcal{M} \) and \( \mathcal{L} \). The functor which assigns to each \((\mathcal{H}om^c(U, V) \setminus \mathcal{G}_{(u|v)})\)-comodule \( \mathcal{M} \) the functor

\[ (Y \setminus \mathcal{G}_v) \longrightarrow \text{Comod} \longrightarrow (X \setminus \mathcal{G}_u) \longrightarrow \text{Comod}, \quad \mathcal{L} \mapsto (\tilde{\mathcal{M}}(\mathcal{L}), \xi_{\tilde{\mathcal{M}}(\mathcal{L})}) \]

is a direct image functor of the morphism (11).

10.3.7.2. The composition of morphisms. Let \( U \xrightarrow{u} X \), \( V \xrightarrow{v} Y \) and \( W \xrightarrow{w} Z \) be affine weakly flat morphisms which allows to assume that \( X = \text{Sp}^\circ(U \setminus \mathcal{G}_u), \)

63
Let \( Y = \text{Sp}^o(V \setminus \mathcal{G}_v) \), and \( Z = \text{Sp}^o(W \setminus \mathcal{G}_w) \). Let \( \mathcal{M} = (M, \zeta_u, \zeta^w) \) be a \( ((U \setminus \mathcal{G}_u), (V \setminus \mathcal{G}_v)) \)-bicomodule and \( \mathcal{N} = (N, \zeta_v, \zeta^w) \) a \( ((V \setminus \mathcal{G}_v), (W \setminus \mathcal{G}_w)) \)-bicomodule. Consider the diagram

\[
\begin{array}{ccc}
MN & \xrightarrow{\zeta^w} & M \mathcal{G}_v N \\
\downarrow{\zeta_v} & & \downarrow{\zeta_v} \\
\end{array}
\]

This diagram is coreflexive, hence by hypothesis it has a kernel, \( M \otimes N \). The coaction \( \zeta^w : N \rightarrow N \mathcal{G}_w \) induces a right \( (W \setminus \mathcal{G}_w) \)-comodule structure, \( M \otimes \zeta^w \) on \( M \otimes N \). Since the functor \( G_u = u^*u_* \) is weakly flat, by a standard argument, the left \( (U \setminus \mathcal{G}_u) \)-comodule structure \( \zeta_u : M \rightarrow G_u M \) induces a left \( (U \setminus \mathcal{G}_u) \)-comodule structure, \( \zeta \otimes N \), on \( M \otimes N \). We denote by \( \mathcal{M} \otimes \mathcal{N} \) the \( ((U \setminus \mathcal{G}_u), (W \setminus \mathcal{G}_w)) \)-bicomodule \( (M \otimes N, \zeta_u \otimes N, M \otimes \zeta^w) \).

The functor

\[
\phi_{\mathcal{M} \otimes \mathcal{N}}^* : (W \setminus \mathcal{G}_w) - \text{Comod} \longrightarrow (U \setminus \mathcal{G}_u) - \text{Comod}
\]

corresponding to \( \mathcal{M} \otimes \mathcal{N} \) is isomorphic to the composition \( \phi_{\mathcal{M}}^* \circ \phi_{\mathcal{N}}^* \) of the functors corresponding to the resp. bicomodules \( \mathcal{N} \) and \( \mathcal{M} \). This follows from general facts on commuting limits and from the fact that the limits of functors are computed object-wise (cf. [GZ], Glossary).
Complementary facts.

C1. Continuous morphisms to the categoric spectrum of a monoid. Let \( \mathcal{M} \) be a monoid. Denote by \( \text{Sp}(\mathcal{M}/\mathcal{E}) \) the object of \( |\text{Cat}|^{\mathcal{O}} \) corresponding to the category \( \mathcal{M} - \text{Sets} \) of \( \mathcal{M} \)-sets. We call \( \text{Sp}(\mathcal{M}/\mathcal{E}) \) the categoric spectrum of the monoid \( \mathcal{M} \).

C1.1. Proposition. Let \( X \xrightarrow{f} \text{Sp}(\mathcal{M}/\mathcal{E}) \) be a continuous morphism. Then

(a) The morphism \( f \) is determined by the object \( \mathcal{O} = f^*(\mathcal{M}) \) uniquely up to isomorphism.

(b) There exists a coproduct of any set of copies of \( \mathcal{O} \).

(c) The object \( \mathcal{O} \) has a structure of an \( \mathcal{M}^\circ \)-module, i.e. there is a monoid morphism \( \mathcal{M} \xrightarrow{\phi_f} C_X(\mathcal{O}, \mathcal{O})^\circ \).

Proof. These facts follow from the canonical isomorphisms

\[
\text{Hom}_X(f^*(\mathcal{M}), -) \simeq \text{Hom}_\mathcal{M}(\mathcal{M}, f_*(-)) \simeq f_*(-).
\]

Since \( f^* \) has a right adjoint, it preserves colimits, in particular coproducts. Notice that the category of \( \mathcal{M} \)-sets has small coproducts and cokernels of pairs of arrows, hence it has arbitrary small colimits. The rest of the argument follows the same lines as the argument of 4.1. Details are left to the reader. \( \blacksquare \)

For any object \( \mathcal{O} \) of the category \( C_X \), we denote by \( \Gamma_X \mathcal{O} \) the monoid \( C_X(\mathcal{O}, \mathcal{O})^\circ \).

We denote by \( |\text{Cat}|^{\mathcal{O}}(\mathcal{M}/\mathcal{E}) \) the category of right \( (\mathcal{M}/\mathcal{E}) \)-modules. Its objects are triples \( (X, \mathcal{O}_X, \phi) \), where \( X \) is an object of \( |\text{Cat}|^{\mathcal{O}} \), \( (\mathcal{O}_X, \phi) \) is a right \( \mathcal{M} \)-module in the category \( C_X \), i.e. \( \phi \) is a monoid morphism \( \mathcal{M} \longrightarrow C_X(\mathcal{O}_X, \mathcal{O}_X)^\circ \) (cf. C1.1(c)). Morphisms from \( (X, \mathcal{O}_X, \phi) \) to \( (Y, \mathcal{O}_Y, \psi) \) are given by morphisms \( X \xrightarrow{f} Y \) such that there exists an isomorphism \( \lambda : f^*(\mathcal{O}_Y) \xrightarrow{\sim} \mathcal{O}_X \) making the diagram

\[
\begin{array}{ccc}
\mathcal{M}^\circ & \xrightarrow{\phi} & C_X(\mathcal{O}_X, \mathcal{O}_X) \\
\psi \downarrow & & \downarrow \mathcal{c}_\lambda \\
C_Y(\mathcal{O}_Y, \mathcal{O}_Y) & \xrightarrow{f_\mathcal{O}_Y \circ \mathcal{O}_Y} & C_X(f^*(\mathcal{O}_Y), f^*(\mathcal{O}_Y))
\end{array}
\]

commute. Here \( \mathcal{c}_\lambda \) denotes the conjugation by \( \lambda, h \mapsto \lambda^{-1} \circ h \circ \lambda \).

Let \( |\text{Cat}|^{\mathcal{O}}(\mathcal{M}/\mathcal{E})_c \) denote the full subcategory of the category \( |\text{Cat}|^{\mathcal{O}}(\mathcal{M}/\mathcal{E}) \) whose objects are triples \( (X, \mathcal{O}_X, \phi) \) such that the following conditions hold:

(a) There exists a coproduct of any set of copies of \( \mathcal{O}_X \).

(b) Let \( \Phi_\phi \) denote the functor from the subcategory \( \mathcal{L}_\mathcal{M} \) of free \( (\mathcal{M}/\mathcal{E}) \)-sets to \( C_X \) which is uniquely defined by the action \( \phi \) (thanks to (b) above). The image by \( \Phi_\phi \) of any pair of arrows \( X_1 \xrightarrow{\gamma} X_0 \) has a cokernel.

Let \( \text{Cat}^{op}(\mathcal{M}/\mathcal{E})_c \) denote the preimage of \( |\text{Cat}|^{\mathcal{O}}(\mathcal{M}/\mathcal{E})_c \) in \( \text{Cat}^{op}(\mathcal{M}/\mathcal{E}) \). On the other hand, let \( (|\text{Cat}|^{\mathcal{O}}/\text{Sp}(\mathcal{M}/\mathcal{E})_c \) denote the full subcategory of the category \( |\text{Cat}|^{\mathcal{O}}/\text{Sp}(\mathcal{M}/\mathcal{E}) \) whose objects are continuous morphisms to \( \text{Sp}(\mathcal{M}/\mathcal{E}) \), and let \( (\text{Cat}^{op}/\text{Sp}(\mathcal{M}/\mathcal{E}))_c \) denote its preimage in \( \text{Cat}^{op}/\text{Sp}(\mathcal{M}/\mathcal{E}) \).
C1.2. Proposition. The functor (3) induces an equivalence of the fibered categories

\[
\begin{pmatrix}
(Cat^{op}/Sp(M/E))_c \\
(Cat^{op}/Sp(M/E))_c
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
(Cat^{op}(M/E))_c \\
(Cat^{op}(M/E))_c
\end{pmatrix}.
\]

(4\text{bis})

Proof. The argument is similar to that of 4.4.2. ■

C1.3. $E$-Spaces. Denote by $|Cat|^o_E$ the category whose objects are pairs $(X, O)$, where $X \in Ob|Cat|^o$ and $O$ is an object of $C_X$ such that there exist coproducts of small sets of copies of $O$ and any pair of arrows $O^\oplus I \longrightarrow O^\oplus J$ between coproducts of copies of $O$ has a cokernel in $C_X$ (equivalently, every small diagram $O : D \longrightarrow C_X$ which maps all objects to $O$ has a colimit in $C_X$). Morphisms from $(X, O)$ to $(X', O')$ are morphisms $X \xrightarrow{f} X'$ such that there exists an isomorphism $f^*(O') \xrightarrow{\sim} O$. Composition is defined in an obvious way. Objects of the category $|Cat|^o_E$ will be called $E$-‘spaces’.

It follows that $E$-‘spaces’ are precisely those pairs $(X, O)$ for which the functor

\[f_* : C_X \longrightarrow \Gamma_X O - Sets, \ M \longmapsto C_X(O, M),\]

is a direct image functor of a continuous morphism $f : X \longrightarrow Sp(\Gamma_X O/E)$; i.e. $f_*$ has a left adjoint.

C1.4. Affine $E$-spaces. We call a $E$-space $(X, O)$ affine if the canonical morphism $f_ : X \longrightarrow Sp(\Gamma_X O)$ is an isomorphism; i.e. the functor $f_*$ (see (6)) is a category equivalence. We denote by $Aff_E$ the full subcategory of the category $|Cat|^o_E$ objects of which are affine $E$-spaces.

Let $Aff^\varepsilon_E$ denote the subcategory of the category $|Cat|^o_E$ whose objects are pairs $(Sp(M/E), M)$. Morphisms from $(Sp(M/E), M)$ to $(Sp(M'/E), M')$ are morphisms $Sp(M/E) \longrightarrow Sp(M'/E)$ corresponding to monoid morphisms $M' \longrightarrow M$. There is an inclusion functor $Aff^\varepsilon_E \xrightarrow{\gamma_*} |Cat|^o_E$ which takes values in the subcategory of affine $E$-spaces.

C1.4.1. Proposition. The functor $Aff^\varepsilon_E \xrightarrow{\gamma_*} |Cat|^o_E$ is fully faithful and has a left adjoint. In particular, $\gamma_*$ induces an equivalence of $Aff^\varepsilon_E$ and the category of affine $E$-spaces.

Proof. The argument is similar to that of 4.5.4 and is left to the reader. ■

C2. Continuous morphisms, monads, and localizations. Let $X \xrightarrow{q} Y$ be a localization with an inverse image functor $C_Y \xrightarrow{q^*} C_X$. Let $\Sigma_q$ denote the class of all morphisms $s$ in $C_Y$ such that $q^*(s)$ is invertible. A functor $C_Y \xrightarrow{F} C_Y$ is compatible with the localization $q$ iff $F(\Sigma_q) \subseteq \Sigma_q$. In this case, there exists a unique functor $C_X \xrightarrow{\bar{F}} C_X$ such that $q^* \circ \bar{F} = \bar{F} \circ q^*$.

C2.1. Proposition. Let $X \xrightarrow{q} Y$ be a continuous localization.

(a) A functor $C_Y \xrightarrow{F} C_Y$ is compatible with $q$ iff the canonical morphism

\[q^* \circ F \longrightarrow q^* \circ F \circ q^* \]

(1)
is an isomorphism.

(b) Suppose $C_Y \xrightarrow{F} C_Y$ is compatible with the localization $q$, and let $\bar{F}$ be the functor $C_X \xrightarrow{\bar{F}} C_X$ such that $q^* \circ F = \bar{F} \circ q^*$.

(i) If $C_Y$ has colimits of certain type, then $C_X$ has colimits of this type. If $F$ preserves colimits of this type, then the functor $\bar{F}$ has the same property.

(ii) If $C_Y$ has limits of certain type, then $C_X$ has limits of this type. If $F$ and $q^*$ preserve limits of this type (e.g. finite limits), then the functor $\bar{F}$ has the same property.

(iii) If $F$ has a right adjoint, then $\bar{F}$ has a right adjoint.

Proof. (a) The assertion follows from C2.4.11 applied to $f^* = q^* \circ F$.

(b) (i) Let $D : D \to C_X$ be a small diagram such that there exists $\colim(q_*D)$. Then there exists the colimit of $D$ and $\colim(D) = q^*(\colim(q_*D))$. Suppose the functor $F$ preserves the colimit of $q_*D$. Since $\bar{F} \simeq q^*Fq_*$, we have:

$$\bar{F}(\colim(D)) \simeq q^*Fq_*(\colim(q_*D)) \simeq q^*(\colim(Fq_*D))$$

(the second isomorphism here is due to the isomorphism $q^*F \xrightarrow{\sim} q^*Fq_*q^*$). Since $q^*$ has a right adjoint, it preserves colimits. Therefore

$$q^*(\colim(Fq_*D)) \simeq \colim(q^*Fq_*D) \simeq \colim(\bar{F}D)$$

whence the assertion.

(ii) Let $D : D \to C_X$ be a small diagram such that there exists $\lim(q_*D)$. Then, by [GZ, 1.1.4], there exists the limit of $D$ and $\lim(D) = q^*(\lim(q_*D))$. Let the functor $F$ preserve the colimit of $q_*D$. As in (i), we have:

$$\bar{F}(\lim(D)) \simeq q^*Fq_*(\lim(q_*D)) \simeq q^*(\lim(Fq_*D))$$

If $q^*$ preserves limit of $Fq_*D$, we continue as follows:

$$q^*(\lim(Fq_*D)) \simeq (\lim(q^*Fq_*D)) \simeq \lim(\bar{F}D).$$

(iii) Let $F^!$ be a right adjoint to $F$. Set $\bar{F}^! = q^*F^!q_*$. By (a), $\bar{F} \simeq q^*Fq_*$. Thus we have morphisms

$$\bar{F} \bar{F}^! \xrightarrow{\sim} (q^*Fq_*q^*)F^!q_* \xrightarrow{\sim} q^*FF^!q_* \xrightarrow{q^*\epsilon_Fq_*} q^*q_* \xrightarrow{\epsilon_q} Id_{C_X}. \quad (3)$$

and

$$Id_{C_X} \xrightarrow{\sim} q^*q_* \xrightarrow{q^*\eta_Fq_*} q^*F^!Fq_* \xrightarrow{q^*F^!\eta_qF_*} q^*F^!q_*q^*Fq_* \xrightarrow{\sim} \bar{F} \bar{F}^! \xrightarrow{\sim} \bar{F} \bar{F}^!. \quad (4)$$

The compositions of the sequence of morphisms resp. (3) and (4) are adjunction arrows.

C2.2. Proposition. Let $X \xrightarrow{q} Y$ be a localization and $F = (F, \mu)$ a monad on $Y$ such that the endofunctor $F$ is compatible with $q$. Then the monad $\mathcal{F}$ induces a monad, $\mathcal{F} = (\bar{F}, \bar{\mu})$, on $X$ defined uniquely up to isomorphism.

(i) If $\mathcal{F}$ is continuous (i.e. $F$ has a right adjoint), then the monad $\bar{F}$ is continuous.
(ii) If $C_y$ has colimits of certain type, then $C_x$ has colimits of this type. If $F$ preserves colimits of this type, then $F$ has the same property.

(iii) If $C_y$ has limits of certain type, then $C_x$ has limits of this type. If $F$ and $q^*$ preserve limits of this type, then $F$ has the same property.

Proof. Fix an inverse image, $q^*$, of the localization $q$. Let $\bar{F}$ be a unique endofunctor $\bar{F} : C_x \rightarrow C_x$ such that $q^* \circ F = \bar{F} \circ q^*$. Then $q^* \circ F^2 = \bar{F}^2 \circ q^*$, and, by the universal property of localizations, there exists a unique morphism $\bar{\mu} : \bar{F}^2 \rightarrow \bar{F}$ such that $q^* \mu = \bar{\mu} q^*$. We leave to the reader verifying that $\bar{\mu}$ is a monad structure on $\bar{F}$.

The assertions (i), (ii), (iii) follow from the corresponding assertions of C2.1. ■

C2.2.1. Remark. The same assertion holds for comonads. In fact, the first part is obtained by dualization. The parts (i) and (ii) are statements about endofunctors.

C3. Cones of non-unital monads and rings.

C3.1. Non-unital monads. Let $X$ be a ‘space’ such that $C_X$ is an additive category, and let $F_+ = (F_+, \mu)$ be a non-unital additive monad on $X$; i.e. $F_+$ is an additive functor $C_X \rightarrow C_X$ and $\mu$ is a functor morphism $F_+^2 \rightarrow F_+$ such that $\mu \circ F_+ \mu = \mu \circ \mu F_+$. Let $F_+ - \text{mod}_1$ denote the category of non-unital $F_+$-modules. Its objects are pairs $(M, \xi)$, where $M \in \text{Ob} C_X$ and $\xi$ a morphism $F_+(M) \rightarrow M$ such that $\xi \circ \mu(M) = \xi \circ F_+ \xi$. A morphism $(M, \xi) \rightarrow (M', \xi')$ is given by a morphism $M \xrightarrow{f} M'$ such that $\xi' \circ F_+(f) = f \circ \xi$. Composition is defined naturally, so that map which assigns to each $F_+$-module $(M, \xi)$ the object $M$ and to every $F_+$-module morphism $(M, \xi) \xrightarrow{f} (M', \xi')$ the morphism $M \xrightarrow{f} M'$ is a functor, $F_+ - \text{mod}_1 \xrightarrow{f} C_X$. This functor has a canonical left adjoint, $f^*$, which maps every object $N$ of $C_X$ to the $F_+$-module $(N \oplus F(N), \xi_N)$, where the action $F_+(N \oplus F_+(N)) = F_+(N) \oplus F_+^2(N)$.

Thus, $f_* f^* = \text{Id}_{C_X} \oplus F$. We denote $\text{Id}_{C_X} \oplus F_+$ by $F = (F, \mu)$. It is easy to see that the category $F_+ - \text{mod}_1$ of non-unital $F_+$-modules is isomorphic to the category $F - \text{mod}$ of unital $F$-modules.

There is a natural embedding $C_X \rightarrow F_+ - \text{mod}$ which assigns to each object $M$ of $C_X$ the $F_+$-module $(M, 0)$. We denote the image of $C_X$ in $F_+ - \text{mod}$ (i.e. the full subcategory generated by trivial modules) by $T_{F_+}$.

C3.2. The cone of a non-unital monad. Suppose that the category $C_X$ is abelian. We denote by $C_{\text{Cone}(F_+/X)}$ the quotient, $F_+ - \text{mod}_1/T_{F_+}$, of the category $F_+ - \text{mod}_1$ by the smallest Serre subcategory containing $T_{F_+}$. This defines a ‘space’ $\text{Cone}(F_+/X)$.

C3.2.1. Proposition. If $F_+$ is a unital monad, then $C_{\text{Cone}(F_+/X)}$ is naturally equivalent to the category $F_+ - \text{mod}$ of unital $F_+$-modules, i.e. the ‘space’ $\text{Cone}(F_+/X)$ is isomorphic to $\text{Sp}(F_+/X)$.

Proof. If the monad $F_+ = (F_+, \mu)$ is unital with the unit element $\text{Id} \xrightarrow{e} F_+$, then there is a monad epimorphism $F = \text{Id} \oplus F_+ \xrightarrow{\gamma} F_+$ defined by $(e, \text{id}_{F_+})$. The corresponding
pull-back functor, $γ_*$, is the inclusion functor $F_+ - mod$ into $F_+ - mod_1$. Its left adjoint, $γ^*$ assigns to each object $(M, ξ)$ of $F_+ - mod_1$ the cokernel of the pair of morphisms $M \xrightarrow{M} M$. Since the functor $γ_*$ is fully faithful, its left adjoint $γ^*$ is an exact localization, and the kernel of $γ^*$ coincides with the subcategory $T_{F_+}$. Therefore, $T_{F_+}$ is, in this case, a Serre subcategory, i.e. $T_{F_+} = T_{F_+}$, whence the assertion. ■

C3.2.2. Corollary. If $F_+$ is a unital monad, then $\text{Sp}(F/X) \simeq \text{Sp}(F_+/X) \coprod X$.

Proof. If $F_+$ is a unital monad, then (by the argument of C3.2.1) $F \simeq F_+ \coprod Id_X$, where $Id_X$ denotes the identical monad $(Id_{C_X}, id)$. This implies that the category $F - mod$ of $F$-modules is equivalent to the product $F_+ - mod \coprod C_X$, hence the assertion. ■

C3.3. The cone of an associative ring. Let $X = \text{Sp}(R)$, where $R_0$ is a unital associative ring, and let $R_+$ be an $R_0$-ring. The latter means that $R_0$ is an associative ring, not unital in general, in the category of $R_0$-bimodules; i.e. the multiplication in $R_+$ is given by an $R_0$-bimodule morphism $R_+ \otimes_{R_0} R_+ \xrightarrow{m} R_+$ satisfying the associativity condition. The $R_0$-ring $R_+$ defines a non-unital monad $F_+ = (F_+, μ)$ on $X$, where $F_+$ is the endofunctor $R_+ \otimes_{R_0} -$ on $C_X = R_0 - mod$ and $F_+^2 = F_+$ is induced by the multiplication $R_+ \otimes_{R_0} R_+ \xrightarrow{m} R_+$. The category $F_+ - mod_1$ of non-unital $F_+$-modules is the category of unital $R_0$-modules endowed with a non-unital $R_+$-module structure compatible with the action of $R_0$ on the module and on $R_+$. We write $R_+ - mod_1$ instead of $F_+ - mod_1$ and $T_{R_+}$ instead of $T_{F_+}$. By definition $T_{F_+}$ is the full subcategory of $R_+ - mod_1$ spanned by modules with zero action.

The associated augmented monad $F$ (cf. C3.1) is isomorphic to the monad associated with the unital $R_0$-ring $R = R_0 \oplus R_+$ which we call the augmented $R_0$-ring corresponding to $R_+$. The category $R_+ - mod_1$ is isomorphic to the category $R - mod$ of unital $R$-modules.

We shall write $\text{Cone}(R_+/R_0)$, or simply $\text{Cone}(R_+)$, instead of $\text{Cone}(F_+/\text{Sp}(R_0))$.

We shall identify $R_+ - mod_1$ with $R - mod$ whenever it is convenient by some reason. Thus, $T_{R_+}$ is viewed as the full subcategory of $R - mod$ whose objects are modules annihilated by the irrelevant ideal $R_+$; and we write $C_{\text{Cone}(R_+)} = R - mod/T_{R_+}$, where $T_{R_+}$ is the smallest Serre subcategory of the category $R - mod$ containing $T_{R_+}$ (getting back the definition of a cone in 1.6). The localization functor $R - mod \xrightarrow{u^*} R - mod/T_{R_+}$ is an inverse image functor of a morphism of ‘spaces’ $\text{Cone}(R_+) \xrightarrow{u} \text{Sp}(R)$. The functor $u^*$ has a (necessarily fully faithful) right adjoint, i.e. the morphism $u$ is continuous. The composition of the morphism $u$ with the natural affine morphism $\text{Sp}(R) \rightarrow \text{Sp}(R_0)$ is a continuous morphism $\text{Cone}(R_+) \rightarrow \text{Sp}(R_0)$. Its direct image functor is (regarded as) the global sections functor.

C3.3.1. Proposition. If $R_+$ is a unital ring, then $\text{Cone}(R_+/R_0) \simeq \text{Sp}(R_+)$ and $\text{Sp}(R) \simeq \text{Sp}(R_+) \coprod \text{Sp}(R_0)$.

Proof. The assertion follows from C3.2.1 and C3.2.2. ■

C3.3.2. Lemma. Let $J$ be a two-sided ideal in the ring $R$ contained in $R_+$ (i.e. a two-sided ideal in $R_+$ which is an $R_0$-bimodule). Let $T_{R_+|J}$ denote the full subcategory of
$R - \text{mod}$ whose objects are $R$-modules annihilated by $\mathcal{J}$; and let $T_{R|\mathcal{J}}^-$ be the Serre subcategory spanned by $T_{R|\mathcal{J}}$. The quotient category $R - \text{mod}/T_{R|\mathcal{J}}^-$ is equivalent to $C_{\text{Cone}(\mathcal{J})}$.

**Proof.** The embedding $\mathcal{J} \hookrightarrow R$ induces a unital ring morphism $\tilde{\mathcal{J}} \rightarrow R$, where $\tilde{\mathcal{J}}$ is the ring $R_0 \oplus \mathcal{J}$ with natural multiplication. The pull-back functor $R - \text{mod} \xrightarrow{\iota_*} \tilde{\mathcal{J}} - \text{mod}$ induces a functor from the subcategory $T_{R|\mathcal{J}}$ to the subcategory $T_{\tilde{\mathcal{J}}}$. Since the functor $\iota_*$ is exact (in a strong sense, that is it preserves small limits and colimits), it maps the Serre subcategory $T_{R|\mathcal{J}}^-$ to the Serre subcategory $T_{\tilde{\mathcal{J}}}^-$. Thus we have a commutative diagram

$$
\begin{array}{ccc}
R - \text{mod} & \xrightarrow{\iota_*} & \tilde{\mathcal{J}} - \text{mod} \\
\uparrow & & \uparrow \\
T_{R|\mathcal{J}}^- & \longrightarrow & T_{\tilde{\mathcal{J}}}^-
\end{array}
$$

of exact functors. Therefore the functor $\iota_*$ induces a functor

$$
R - \text{mod}/T_{R|\mathcal{J}}^- \longrightarrow \tilde{\mathcal{J}} - \text{mod}/T_{\tilde{\mathcal{J}}}^-.
$$

The functor (2) is a category equivalence. In fact, let $\tilde{\mathcal{J}} - \text{mod} \xrightarrow{\Psi} R - \text{mod}$ be the functor which assigns to every $\tilde{\mathcal{J}}$-module $M$ the $R$-module $\mathcal{J}M$. The cokernel of the embedding $\mathcal{J}M \hookrightarrow M$ belongs to $T_{\mathcal{J}}$, hence the localization $\tilde{\mathcal{J}} - \text{mod} \longrightarrow \tilde{\mathcal{J}} - \text{mod}/T_{\tilde{\mathcal{J}}}^-$ maps this embedding to an isomorphism. We assign to each object $M$ of $\tilde{\mathcal{J}} - \text{mod}$ the composition of the functor $\Psi$ and the localization $R - \text{mod} \longrightarrow R - \text{mod}/T_{R|\mathcal{J}}^-$. It follows that this functor factors through the localization $\tilde{\mathcal{J}} - \text{mod} \longrightarrow \tilde{\mathcal{J}} - \text{mod}/T_{\tilde{\mathcal{J}}}^-$, i.e. it defines (uniquely) a functor $\tilde{\mathcal{J}} - \text{mod}/T_{\tilde{\mathcal{J}}}^- \longrightarrow R - \text{mod}/T_{R|\mathcal{J}}^-$. The functor $\Phi$ is a quasi-inverse to the functor (2). □

**C3.3.3. Example: quasi-affine schemes.** Quasi-affine schemes are defined (in [EGA II, 5.1.1]) as open quasi-compact subschemes of affine schemes. Open subschemes of $\text{Spec} A$ are in one-to-one correspondence with the radical ideals in $A$. Quasi-compactness of an open set defined by an ideal $\mathcal{J}$ means that $\mathcal{J}$ is the radical of its finitely generated subideal (this holds in noncommutative case too, see [R, I.5.6]. One can show that the category of quasi-coherent sheaves on the open subscheme of $\text{Spec} A$ defined by the ideal $\mathcal{J}$ is equivalent to the quotient category $A - \text{mod}/T_{A|\mathcal{J}}^-$. By C3.3.2, the latter category is equivalent to the category $C_{\text{Cone}(\mathcal{J})} = \tilde{\mathcal{J}} - \text{mod}/T_{\tilde{\mathcal{J}}}^-$ of modules on the cone $\text{Cone}(\mathcal{J})$ of the (non-unital) $R_0$-ring $\mathcal{J}$.

**C3.3.4. Functoriality.** Let $R_0 - \text{Rings}$ denote the category of (not necessarily unital) $R_0$-rings. A morphism of such rings, $R_+ \xrightarrow{\phi} S_+$, is an $R_0$-bimodule morphism compatible with multiplication. The morphism $\phi$ induces the pull-back functor

$$
S_+ - \text{mod} \xrightarrow{\phi_*} R_+ - \text{mod}
$$

which maps the subcategory $T_{S_+}$ of trivial $S_+$-modules to the category $T_{R_+}$ of trivial $R_+$-modules. Since $\phi_*$ is exact and preserves small colimits, it maps the Serre subcategory
$T_{S^-}$ spanned by $T_{S^+}$ to the Serre subcategory $T_{R^+}^{-}$ spanned by $T_{R^+}$. Therefore, $\phi_*$ induces a unique functor

$$S_+ - \text{mod}_1/T_{S^+}^- \xrightarrow{\phi_*} R_+ - \text{mod}_1/T_{R^+}^-$$

such that the diagram

$$\begin{array}{ccc}
S_+ - \text{mod}_1/T_{S^+}^- & \xrightarrow{\phi_*} & R_+ - \text{mod}_1/T_{R^+}^- \\
q_*^s & \uparrow & q_*^r \\
S_+ - \text{mod}_1 & \xrightarrow{\phi_*} & R_+ - \text{mod}_1
\end{array}$$

(4)

commutes. In general, the functor $\bar{\phi}_*$ does not have a left adjoint, hence it cannot be interpreted as a direct image functor of a continuous morphism.

**C3.3.4.1. The category $R_0 - \text{Rings}_1$.** We denote by $R_0 - \text{Rings}_1$ the subcategory of $R_0 - \text{Rings}$ formed by $R_0$-ring morphisms $R_+ \xrightarrow{\phi} S_+$ whose inverse image functor, $R_+ - \text{mod}_1 \longrightarrow S_+ - \text{mod}_1$, is compatible with the localizations at resp. $T_{R^+}^-$ and $T_{S^+}^-$. The compatibility means that there exists a functor $C_{\text{Cone}(R_+)} \xrightarrow{\bar{\phi}_*} C_{\text{Cone}(S_+)}$ such that the diagram

$$\begin{array}{ccc}
C_{\text{Cone}(R_+)} = R_+ - \text{mod}_1/T_{R^+}^- & \xrightarrow{\bar{\phi}_*} & S_+ - \text{mod}_1/T_{S^+}^- = C_{\text{Cone}(S_+)} \\
q_*^r & \uparrow & q_*^s \\
R_+ - \text{mod}_1 & \xrightarrow{\phi_*} & S_+ - \text{mod}_1
\end{array}$$

(5)

commutes. Thanks to the universal property of localizations, the functor $\bar{\phi}_*$ is uniquely determined by the commutativity of (5).

Evidently, all ring isomorphisms belong to $R_0 - \text{Rings}_1$. It follows from the universal property of localizations that the composition of morphisms of $R_0 - \text{Rings}_1$ belongs to $R_0 - \text{Rings}_1$; i.e. $R_0 - \text{Rings}_1$ is, indeed, a subcategory of the category $R_0 - \text{Rings}$. The map $R_+ \longmapsto \text{Cone}(R_+)$ extends to a functor $R_0 - \text{Rings}_1^{op} \longrightarrow |\text{Cat}|^o$ which we denote by $\text{Cone}$.

**C3.3.4.2. Remarks.** (a) For any morphism $R_+ \xrightarrow{\varphi} S_+$ of $R_0 - \text{Rings}$, the functor

$$\bar{\varphi}_* = q_*^s \varphi_1 q_*^r : C_{\text{Cone}(R_+)} \longrightarrow C_{\text{Cone}(S_+)}$$

(6)

might be regarded as an inverse image functor of a morphism $\text{Cone}(S_+) \xrightarrow{\bar{\varphi}} \text{Cone}(R_+)$. Notice, however, that the map $\varphi \longmapsto \bar{\varphi}$ is not functorial, unless morphisms are picked from the subcategory $R_0 - \text{Rings}_1$.

(b) For any morphism $R_+ \xrightarrow{\varphi} S_+$ of $R_0 - \text{Rings}_1$, the corresponding morphism $\text{Cone}(S_+) \xrightarrow{\bar{\varphi}} \text{Cone}(R_+)$ is continuous.
This follows from the fact that the functor $C_{\text{Cone}(R_+)} \xrightarrow{q_{R^*}} R_+ \text{ mod } 1$ has a right adjoint and from the formula (6).

**C3.3.4.3. Proposition.** Let $S_+$ be an $R_0$-ring, $e$ a central idempotent element in $S_+$ (i.e. $e^2 = e$). Then $R_+ = \{ r \in S \mid re = er = r \}$ is an $R_0$-subring in $S_+$, and the inclusion $R \hookrightarrow S$ is a morphism of $R_0 - \text{Rings}_1$.

**Proof** is left to the reader. □

**C3.3.5. Remark.** For any $R_0$-ring $R_+$, we have a canonical morphism (Zariski open immersion) $\text{Cone}(R_+) \rightarrow \text{Sp}(R)$, $R = R_0 \oplus R_+$, which depends functorially on $R_+$ (a functor from $R_0 - \text{Rings}_{1}^{gr}$). This morphism can be regarded as a noncommutative analogue of the Stone compactification of a locally compact space. If the ring $R_+$ is unital, then $\text{Sp}(R)$ is the disjoint union of $\text{Sp}(R_+)$ and $\text{Sp}(R_0)$ (see C3.3).

**C3.3.6. Hopf actions and cross-products.** Let $R_0$ be an associative unital $k$-algebra. We call an $R_0$-ring $R_+$ an $(R_0 | k)$-ring if the $R_0$-ring structure makes $R_+$ a $k$-algebra, i.e. $\lambda r = r \lambda$ for all $r \in R_+$ and $\lambda \in k$. Let $\mathcal{H} = (\delta, H, \mu)$ be a $k$-bialgebra. Here $H \xrightarrow{\delta} H \otimes_k H \xrightarrow{\mu} H$ are resp. comultiplication and multiplication. Recall that a Hopf action of $\mathcal{H}$ on a $k$-algebra $R_+$ is a unital $H$-module structure on $R$ such that the multiplication $R_+ \otimes_k R_+ \rightarrow R_+$ is an $\mathcal{H}$-module morphism. We assume that $\mathcal{H}$ acts trivially on $R_0$. Then the cross-product $R_+ \# \mathcal{H}$ is an $(R_0 | k)$-ring.

The Hopf action of $\mathcal{H}$ on $R_+$ induces an endofunctor, $\tilde{H}$, on the category $R_+ \text{ mod } 1$. This endofunctor assigns to any (non-unital) $R_+$-module $\mathcal{M} = (M, R_+ \otimes_k M \xrightarrow{\xi} M)$ the $R_+$-module $\mathcal{H} \otimes_k \mathcal{M} = (H \otimes_k M, \xi_H)$, where the action $\xi_H$ is the composition of

$$R_+ \otimes_k H \otimes_k M \xrightarrow{\sim} H \otimes_k R_+ \otimes_k M \rightarrow H \otimes_k H \otimes_k R_+ \otimes_k M \rightarrow H \otimes_k R_+ \otimes_k M \rightarrow H \otimes_k M.$$

Here the second arrow is induced by the comultiplication $\delta$, the third arrow by the action $\tau$, and the fourth arrow by the $R$-module structure $\xi$. The multiplication $H \otimes_k H \xrightarrow{\mu} H$ induces a monad structure, $\tilde{H}^2 \xrightarrow{\mu} \tilde{H}$, on $\tilde{H}$. One can see that the category $R_+ \# \mathcal{H} \text{ mod } 1$ is isomorphic to the category $\tilde{H} \text{ mod } 1$, where $\tilde{H}$ denotes the monad $(\tilde{H}, \tilde{\mu})$. This follows from the observation that the functor $\tilde{H}$ is isomorphic to $R \# \mathcal{H} \otimes R$, where $R = R_0 \oplus R_+$ is the augmented $R_0$-ring. This observation implies, on the other hand, that the functor $\tilde{H}$ is continuous (i.e. it has a right adjoint) and that there is a natural isomorphism between the category $R_+ \# \mathcal{H} \text{ mod } 1$ and the category $(\tilde{H} / R) \text{ mod } 1$ of modules over the monad $\tilde{H}$. Here we write $(\tilde{H} / R)$ instead of $(\tilde{H} / \text{Sp}(R))$ and identify $R_+ \text{ mod } 1$ with $R \text{ mod } 1$. Thus, we have a natural isomorphism $\text{Sp}(\tilde{H} / R) \xrightarrow{\sim} \text{Sp}(R \# \mathcal{H})$ such that the diagram

$$\text{Sp}(\tilde{H} / R) \xrightarrow{\sim} \text{Sp}(R \# \mathcal{H}) \xrightarrow{\sim} \text{Sp}(R)$$

commutes.

The following assertion provides another family of morphisms of $R_0 - \text{Rings}_1$. 72
C3.3.6.1. Proposition. Let $H \otimes_k R_+ \xrightarrow{\tau} R_+$ be a Hopf action of an $k$-bialgebra $H = (\delta, H, \mu)$ on a $(R_0|k)$-ring $R_+$. Suppose the functor $H \otimes_k -$ is flat. Then the monad $\mathcal{H}$ on $\text{Sp}(R)$ induces a monad $\mathcal{H}$ on $\text{Cone}(R_+)$ such that there is a canonical commutative diagram

$$\begin{array}{ccc}
\text{Sp}(\mathcal{H}/\text{Cone}(R_+)) & \xrightarrow{\sim} & \text{Cone}(R_+\#\mathcal{H}) \\
\downarrow & & \downarrow \\
\text{Cone}(R_+) & & 
\end{array}$$

of affine morphisms.

In particular, the canonical morphism $R_+ \rightarrow R_+\#\mathcal{H}$ belongs to $R_0 - \text{Rings}_1$.

Proof. It follows that the functor $\tilde{H}$ maps the subcategory $T_{R_+}$ to itself. Since the functor $H \otimes_k - : k - \text{mod} \rightarrow k - \text{mod}$ is flat (i.e. it is exact and preserves colimits of small diagrams), the functor $R_+ - \text{mod}_1 \xrightarrow{\tilde{H}} R_+ - \text{mod}_1$ is flat too. Therefore, the Serre subcategory $T_{R_+}$ is stable under $\tilde{H}$, and the functor $\tilde{H}$ induces a continuous functor $R_+ - \text{mod}_1/T_{R_+} = C_{\text{Cone}(R_+)} \xrightarrow{\tilde{H}} C_{\text{Cone}(R_+)}$. By C2.2, the multiplication $\tilde{H}^2 \xrightarrow{\mu} \tilde{H}$ induces a multiplication $\tilde{H}^2 \xrightarrow{\mu} \tilde{H}$. The isomorphism of categories

$$R_+\#\mathcal{H} - \text{mod}_1 \xrightarrow{\sim} (\mathcal{H}/R) - \text{mod}$$

mentioned above induces an isomorphism $C_{\text{Cone}(R\#\mathcal{H})} \xrightarrow{\sim} (\mathcal{H}/\text{Cone}(R)) - \text{mod}$, regarded as an inverse image functor of an isomorphism $\text{Sp}(\mathcal{H}/\text{Cone}(R)) \xrightarrow{\sim} \text{Cone}(R\#\mathcal{H})$ such that the diagram (8) commutes. The monad $(\mathcal{H}/R)$ is continuous (i.e. the functor $\tilde{H}$ has a right adjoint), because $\tilde{H}$ is isomorphic to the (obviously) continuous functor $R\#\mathcal{H} \otimes_R R$.

By C2.2(i), this implies that the monad $\mathcal{H}$ on $\text{Cone}(R)$ is continuous. By 6.2, the latter means precisely that the natural morphism $\text{Sp}(\mathcal{H}/\text{Cone}(R)) \rightarrow \text{Cone}(R)$ is affine. Therefore, by the commutativity of (8), the morphism $\text{Cone}(R\#\mathcal{H}) \rightarrow \text{Cone}(R)$ is affine. This shows, in particular, the canonical morphism $R_+ \rightarrow R_+\#\mathcal{H}$ belongs to $R_0 - \text{Rings}_1$. ■

C4. Noncommutative projective spectra.

C4.1. Proj. Fix a monoid $G$. Let $F_+ = (F_+, \mu)$ be a $G$-graded (non-unital in general) monoid on $X$. Let $\text{gr}_G F_+ - \text{mod}_1$ denote the category of $G$-graded non-unital $F_+$-modules and preserving gradings morphisms. Let

$$\text{gr}_G F_+ - \text{mod}_1 \xrightarrow{\pi^*} F_+ - \text{mod}_1$$

be the functor forgetting the grading. We denote by $\text{gr}_G T_{F_+}$ the preimage of the subcategory $T_{F_+}$ in $\text{gr}_G F_+ - \text{mod}_1$. Let $\text{Proj}_G(F_+) = \text{Proj}_G(F_+)$ be the quotient category $\text{gr}_G F_+ - \text{mod}_1/\text{gr}_G T_{F_+}$. This defines a 'space' $\text{Proj}_G(F_+) = \text{Proj}_G(F_+/X)$.

C4.2. Actions. Let $G$ be a monoid. An action of $G$ on a 'space' $X$ is a monoidal functor $G \xrightarrow{\mathcal{G}} \text{End}(C_X)$. Here $G$ is viewed as a discrete monoidal category and $\text{End}(C_X)$...
denote the (strict) monoidal category of endofunctors $C_X \rightarrow C_X$; i.e. $\widehat{\operatorname{End}}(C_X) = (\operatorname{End}(C_X), \circ)$.

**C4.2.1. Examples.** (a) Let $\mathbb{F}_+$ be a (non-unital in general) $\mathcal{G}$-graded monad on a ‘space’ $X$; and let $C_Y = \text{gr}_G \mathbb{F}_+ - \text{mod}_1$. For any $\mathcal{G}$-graded $\mathbb{F}_+$-module $N = \oplus_{v \in \mathcal{G}} N_v$ and any $\gamma \in \mathcal{G}$, we denote by $N[\gamma]$ the $\mathcal{G}$-graded $\mathbb{F}_+$-module defined by $N[\gamma]_\sigma = N_{\sigma \gamma}$. This defines a strict action of $\mathcal{G}$ on the ‘space’ $Y$. Here strict means that the monoidal functor $\mathcal{G} \rightarrow \widehat{\operatorname{End}}(C_X)$ is strict, that is $N[\gamma_1 \gamma_2] = (N[\gamma_2])[\gamma_1]$ for all $N$.

(b) The action of $\mathcal{G}$ on the ‘space’ $Y$ in (a) (i.e. on the category $\text{gr}_G \mathbb{F}_+ - \text{mod}_1$), induces an action of $\mathcal{G}$ on $\text{Proj}(\mathbb{F}_+)$. 

**C4.3. Proposition.** Let $\mathbb{T}$ be a $\mathcal{G}$-stable, topologizing subcategory of $\text{gr}_G \mathbb{F}_+ - \text{mod}_1$, and let $\mathbb{T}^-$ denote the image of $\mathbb{T}$ in $\mathbb{F}_+ - \text{mod}_1$. Then $\pi^\ast - (\mathbb{T}^-) \subseteq \mathbb{T}^-.$

If the ‘space’ $X$ has the property (sup) and the functor $F_+$ preserves supremsums of subobjects, then $\pi^\ast - (\mathbb{T}^-) = \mathbb{T}^-.$

**Proof.** (a) Since the functor (1) is exact, the preimage, $\pi^\ast - (\mathbb{T}^-)$, of the Serre subcategory $\mathbb{T}^-$ is a Serre subcategory of the category $\text{gr}_G \mathbb{F}_+ - \text{mod}_1$. The inclusion $\pi^\ast - (\mathbb{T}^-) \subseteq \mathbb{T}^-$ is equivalent to that every nonzero object of $\pi^\ast - (\mathbb{T}^-)$ has a nonzero subobject which belongs to $\mathbb{T}$; or, what is the same, for any nonzero object, $N$, of $\pi^\ast - (\mathbb{T}^-)$, there exists a nonzero morphism $L \rightarrow N$, with $L \in \text{Ob} \mathbb{T}$. We can and will assume that $L$ is generated by one of its homogeneous components. Then $\mathbb{F}_+ - \text{mod}_1(L, N)$ is a $\mathcal{G}$-graded $\mathbb{Z}$-module, and some of homogeneous components of the morphism $g$ are nonzero. Replacing the module $L$ by the module $L[\gamma]$ for an appropriate $\gamma \in \mathcal{G}$, we can assume that the homogeneous component of $g$ of zero degree is nonzero. Thus, there exists a nonzero morphism $L[\gamma] \rightarrow N$ of graded $\mathbb{F}_+$-modules. Since the subcategory $\mathbb{T}$ is stable under the action of $\mathcal{G}$, the object $L[\gamma]$ belongs to $\mathbb{T}$.

(b) Since $X$ has the property (sup) and $F$ preserves supremsums, both categories, $\text{gr}_G \mathbb{F}_+ - \text{mod}_1$ and $\mathbb{F}_+ - \text{mod}_1$ possess this property too. Therefore, every object, $M$, of $\mathbb{T}^-$ has a filtration, $\{M_i \mid i \geq 0\}$ such that $M_i = \text{sup}(M_j \mid j < i)$, if $i$ is a limit ordinal, and $M_{i+1}/M_i$ belongs to $\mathbb{T}$. But, this implies that $M$ is an object of $\mathbb{T}^-$; i.e. we have the inverse inclusion, $\mathbb{T}^- \subseteq \pi^\ast - (\mathbb{F}_+)$. \[\blacksquare\]

**C4.3.1. Corollary.** (a) $\pi^\ast - (\mathbb{T}_{\mathbb{F}_+}^-) \subseteq \text{gr}_G \mathbb{T}_{\mathbb{F}_+}^-.$

(b) If the ‘space’ $X$ has the property (sup) and the functor $F_+$ preserves supremsums of subobjects, then $\text{gr}_G \mathbb{T}_{\mathbb{F}_+}^- = \pi^\ast - (\mathbb{T}_{\mathbb{F}_+}^-).$

**Proof.** Set $\mathbb{T} = \text{gr}_G \mathbb{T}_{\mathbb{F}_+}$. Then $\mathbb{T}^-$ coincides with $\mathbb{T}_{\mathbb{F}_+}^-$, hence the assertion. \[\blacksquare\]

**C4.3.2. Corollary.** Suppose that $X$ has the property (sup) and $F_+$ preserves supremsums of subobjects. Then the forgetful functor (1) induces a faithful exact functor

$$C_{\text{Proj}(\mathbb{F}_+)} \xrightarrow{p^*} C_{\text{Cone}(\mathbb{F}_+)} \quad (2)$$

74
Proof. By C4.3.1(b), \( gr \mathcal{G} T^{-}_{R_+} = \pi^{-1} (T_{+}^{-}) \), where \( gr \mathcal{G} F_+ - \text{mod}_1 \xrightarrow{\pi^{-}} F_+ - \text{mod}_1 \) is the forgetful functor. The functor \( \pi^{-1} \) induces a faithful functor between quotient categories \( gr \mathcal{G} F_+ - \text{mod}_1 / gr \mathcal{G} T_{R_+}^{-} \rightarrow F_+ - \text{mod}_1 / T_{R_+}^{-} \). This functor is exact because the inclusion functor \( gr \mathcal{G} F_+ - \text{mod}_1 \rightarrow F_+ - \text{mod}_1 \) is exact. Hence the assertion. ■

The functor (2) is regarded as an inverse image functor of a morphism ('projection') \( \text{Cone}(F_+) \xrightarrow{p} \text{Proj}_G(F_+) \).

C4.3.3. The Proj of an associative ring. Let \( R_0 \) be an associative unital ring and \( G \) a monoid. Let \( R_+ \) be a \( G \)-graded \( R_0 \)-ring, which is, by definition, a \( G \)-graded ring in the category of \( R_0 \)-bimodules. Then we have the category \( gr \mathcal{G} R_+ - \text{mod}_1 \) of \( G \)-graded \( R_+ \)-modules and its subcategory \( gr \mathcal{G} T_{R_+} = T_{R_+} \cap gr \mathcal{G} R_+ - \text{mod}_1 \). We obtain the 'space' \( \text{Proj}_G(R_+) \equiv \text{Proj}_G(R) \) defined by

\[
C_{\text{Proj}_G(R_+)} = gr \mathcal{G} R_+ - \text{mod}_1 / gr \mathcal{G} T_{R_+}^{-}.
\]

Since the conditions of C4.3.1(b) hold, \( gr \mathcal{G} T_{R_+}^{-} = gr \mathcal{G} R_+ - \text{mod}_1 \cap T_{R_+}^{-} \), and, therefore, we have a canonical projection \( \text{Cone}(R_+) \xrightarrow{p} \text{Proj}_G(R_+) \).

Taking \( X = \text{Sp}(R_0) \) (i.e. \( C_X = R_0 - \text{mod} \)), we can identify \( R_+ \) with the monad \( F_+ = (F_+, \mu) \), where \( F_+ = R_+ \otimes R_0 \) and \( F_+^2 \xrightarrow{\mu} F_+ \) is determined by the multiplication \( R_+ \otimes_{R_0} R_+ \xrightarrow{m} R_+ \). If \( R_+ \) is \( G \)-graded, then the monad \( F_+ \) is \( G \)-graded. We have natural isomorphisms \( \text{Cone}(R_+) \cong \text{Cone}(F_+) \) and \( \text{Proj}_G(R_+) \cong \text{Proj}_G(F_+) \).

We have recovered the construction 1.7 illustrated by examples 1.8, 1.9, and 1.10. One can approach these examples from a different side, via Hopf actions.

C5. Hopf actions. Let \( G \) be a monoid and \( R_0 \) an associative unital \( k \)-algebra. For a \( G \)-graded \( (R_0|k) \)-ring \( R_+ \), we denote by \( gr \mathcal{G} R_+ - \text{mod}_1 \) the category of non-unital \( G \)-graded \( R_+ \)-modules. Let \( \mathcal{H} = (\delta, H, \mu) \) be a \( G \)-graded \( k \)-bialgebra with comultiplication \( \delta \) and multiplication \( \mu \); and let \( \mathcal{H} \otimes_k R_+ \xrightarrow{\tau} R_+ \) is a Hopf action compatible with grading. Recall that a Hopf action of \( \mathcal{H} \) on a \( k \)-algebra \( R_+ \) is a unital \( \mathcal{H} \)-module structure on \( R_+ \) such that the multiplication \( R_+ \otimes_k R_+ \rightarrow R_+ \) is an \( \mathcal{H} \)-module morphism. We assume that \( \mathcal{H} \) acts trivially on \( R_0 \). Then the cross-product \( R_+ \# \mathcal{H} \) is a \( G \)-graded \( (R_0|k) \)-ring.

The Hopf action of \( \mathcal{H} \) on \( R_+ \) induces an endofunctor, \( H_G \), on the category \( gr \mathcal{G} R_+ - \text{mod}_1 \) which assigns to any (non-unital) \( G \)-graded \( R_+ \)-module \( M = (M, R_+ \otimes_k M \xrightarrow{\xi} M) \) the \( G \)-graded \( R_+ \)-module \( \mathcal{H} \otimes_k M = (\mathcal{H} \otimes_k M, \xi_\mathcal{H}) \), where the action \( \xi_\mathcal{H} \) is same as in the non-graded case (cf. C3.3.6). The multiplication \( H \otimes_k H \xrightarrow{\mu} H \) gives rise to a monad \( H_G = (H_G, \mu_G) \) (like in C3.3.6); and the category \( gr \mathcal{G} R_+ \# \mathcal{H} - \text{mod}_1 \) is isomorphic to the category \( \mathcal{H}G - \text{mod} \). By an argument similar to that of C3.3.6, the monad \( H_G \) is continuous.
(i.e. the functor \( H \) has a right adjoint) which is equivalent to that the forgetful functor \( \mathcal{H} - \text{mod} \to \text{gr}_G R_+ - \text{mod} \) is a direct image functor of an affine morphism.

**C5.1. Proposition.** Let \( H \otimes_k R_+ \to R_+ \) be a Hopf action of an \( k \)-bialgebra \( \mathcal{H} = (\delta, H, \mu) \) on a \( \mathcal{G} \)-graded \( (R_0|k) \)-ring \( R_+ \). Suppose the functor \( H \otimes_k - \) is flat. Then the monad \( H \) on \( \text{Sp}(R) \) induces a monad \( \mathcal{H} \) on \( \text{Proj}_G(R_+) \) such that there is a canonical commutative diagram

\[
\text{Sp}(\mathcal{H}/\text{Proj}_G(R_+)) \to \text{Proj}_G(R_+\#\mathcal{H})
\]

of affine morphisms.

**Proof.** The argument is similar to that of C3.3.6.1. Details are left to the reader. ■

**C5.2. Example.** Let \( G \) be a connected reductive algebraic group over an algebraically closed field \( k \) of zero characteristic. Fix a Borel subgroup \( B \), a maximal unipotent subgroup \( U \), and a maximal torus \( H \) chosen in a compatible way: \( H \) and \( U \) are subgroups of \( B \), and \( B = HU \). Let \( R \) be the algebra of regular functions on the homogeneous space \( G/U \) (called after I. M. Gelfand the 'base affine space'). The algebra \( R \) is the direct sum of all simple finite dimensional modules, each appears once; i.e. \( R = \bigoplus_{\lambda \geq 0} R_\lambda \), where \( \lambda \) runs through nonnegative integral weights. Then \( R_0 = k \), and \( R_+ = \bigoplus_{\lambda > 0} R_\lambda \) is a \( \mathcal{G} \)-graded \( k \)-algebra. Here \( \mathcal{G} \) is the group of intergral weights of the group \( G \).

The category \( C_{\text{Cone}(R_+)} \) is equivalent to the category of quasi-coherent sheaves on the base affine space \( G/U \). The category \( C_{\text{Proj}_G(R_+)} \) is equivalent to the category of quasi-coherent sheaves on the flag variety \( G/B \). We refer for details to [LR4].

**C5.2.1. Note.** If the group \( G \) is simply connected, this construction can be given in terms of the Lie algebra \( \mathfrak{g} \) of \( G \) and its Cartan subalgebra \( \mathfrak{h} \), as it is done in 1.9.

**C5.2.2. D-modules.** By construction, there is a Hopf action on \( R \) of the universal enveloping (Hopf) algebra \( U(\mathfrak{g}) \). Consider instead of \( R \) the crossed product \( R_+ \# U(\mathfrak{g}) \).

The universal enveloping algebra, \( \mathcal{H} \), of the Cartan subalgebra, \( \mathfrak{h} \), acts on the algebra \( R \) according the decomposition \( R = \bigoplus_{\lambda > 0} R_\lambda \); each \( R_\lambda \) is a one-dimensional representation of \( \mathcal{H} \) with the weight \( \lambda \) tensored by the vector space \( R_\lambda \). This is a Hopf action commuting with the action of \( U(\mathfrak{g}) \), hence it determines to a Hopf action of \( \tilde{U}(\mathfrak{g}) = U(\mathfrak{g}) \otimes_k \mathcal{H} \) on \( R_+ \).

The category \( C_{\text{Cone}(R_+\#\tilde{U}(\mathfrak{g}))} \) is equivalent to the category of \( D \)-modules on the base affine space \( G/U \).

The category \( C_{\text{Proj}_G(R_+\#\tilde{U}(\mathfrak{g}))} \) is equivalent to the category of \( D \)-modules on the flag variety \( G/B \).

We can express these facts saying that the category of \( D \)-modules on the base affine space \( G/U \) is the category of quasi-coherent sheaves on the noncommutative quasi-affine 'space' \( \text{Cone}(R_+\#\tilde{U}(\mathfrak{g})) \) and the category of \( D \)-modules on the flag variety \( G/B \) is the category of quasi-coherent sheaves on the noncommutative 'space' \( \text{Proj}_G(R_+\#\tilde{U}(\mathfrak{g})) \). Both are semiseparated (actually, separated) noncommutative schemes.
C5.3. Example: quantum affine base space and quantum flag variety. Let $U_q(g)$ be the quantized enveloping algebra of a semisimple Lie algebra, $g$, and let $H$ be its maximal torus (this time canonical). We define $R$ and $R_+$ as in C5.2; i.e. $R = \oplus_{\lambda \geq 0} R_{\lambda}$ and $R_+ = \oplus_{\lambda \geq 0} R_{\lambda}$, where $R_{\lambda}$ is the simple $U_q(g)$-module with the highest weight $\lambda$. The multiplication is given by choosing projections $R_{\lambda} \otimes R_\mu \rightarrow R_{\lambda + \mu}$ for different $\lambda$ and $\mu$ in an appropriate way (see [LR4] for details).

We define the quantum base affine space of $g$ as the 'space' $\text{Cone}(R_+)$ and the quantum flag variety of $g$ as the 'space' $\text{Proj}_g(R_+)$. 

C5.3.1. D-modules on the quantum base affine space and the quantum flag variety. Let $U_q(g)$ be the quantized enveloping algebra of a semisimple Lie algebra, $g$, and let $H$ be its maximal torus. Let $R = \oplus_{\lambda \geq 0} R_{\lambda}$ be the algebra of functions on the quantum base affine space, and $R_+ = \oplus_{\lambda \geq 0} R_{\lambda}$ the quantum base affine space of $g$ which is by definition the spectrum $\text{Cone}(R_+)$ of the algebra $R_+$ (see C5.2).

The maximal torus $H$ acts on $R_+$, and this action commutes with the action of $U_q(g)$. Thus $R_+$ has a structure of a $\tilde{U}_q(g)$-module, where $\tilde{U}_q(g) = U_q(g) \otimes_k H$. By C2.2, $\tilde{U}_q(g)$ induces a continuous monad, $U_q^\sim(g)$, on $\text{Cone}(R_+)$. And we have the commutative diagram

$$\text{Sp}(U_q^\sim(g)/\text{Cone}(R_+)) \rightarrow \text{Cone}(R_+ \# \tilde{U}_q(g)) \rightarrow \text{Cone}(R_+)$$

The action of $\tilde{U}_q(g)$ on $R_+$ respects $G$-grading, hence it induces a continuous monad, $\tilde{U}_q(g)$, on $\text{Proj}_G(R_+)$; and we have a commutative diagram

$$\text{Sp}(\tilde{U}_q(g)/\text{Proj}_G(R_+)) \rightarrow \text{Proj}_G(R_+ \# \tilde{U}_q(g)) \rightarrow \text{Proj}_G(R_+)$$

It is shown in [LR4] that the monad $\tilde{U}_q(g)$ on $\text{Proj}_G(R_+)$ is compatible with the affine localizations

$$\text{Sp}((S_w^{-1}R)_0) \rightarrow \text{Proj}_G(R_+), \quad w \in W,$$

of the quantum flag variety described in 1.10.1. Similarly, the monad $U_q^\sim(g)$ on $\text{Cone}(R_+)$ is compatible with the affine localizations

$$\text{Sp}(S_w^{-1}R) \rightarrow \text{Cone}(R_+), \quad w \in W,$$

of the quantum base affine 'space' (see 1.10.1).

Applying C2.2, we obtain that the affine cover (1) of the quantized flag variety, $\text{Proj}_G(R_+)$, induces an affine cover of $\text{Proj}_G(R_+ \# \tilde{U}_q(g))$.

Similarly, the affine cover (1) of the quantum base affine space $\text{Cone}(R_+)$ induces an affine cover of the 'space' $\text{Cone}(R_+ \# \tilde{U}_q(g))$.

Moreover, it follows from C2.2 that in both cases the morphisms of affine cover are affine localizations. Therefore, $\text{Proj}_G(R_+ \# \tilde{U}_q(g))$ and $\text{Cone}(R_+ \# \tilde{U}_q(g))$ are examples of semiseparated noncommutative D-schemes.
C6. The category of coalgebras and the category of flat, affine covers.

C6.1. The category of coalgebras. Let $R$ be an associative, unital ring, $\mathcal{H} = (H, \delta)$ a coalgebra in the category of $R$-bimodules, $\phi : R \longrightarrow S$ a ring morphism. The comultiplication $\delta$ induces a comultiplication, $\delta_\phi : H_\phi \longrightarrow H_\phi \otimes_S H_\phi$, on the $S$-bimodule $H_\phi = S \otimes_R H \otimes_R S$. In fact, $H_\phi \otimes_S H_\phi \simeq S \otimes_R H \otimes_R S \otimes_R H \otimes_R S$, and the comultiplication $\delta_\phi$ is determined by the composition of $\delta : H \longrightarrow H \otimes_R H$ and $H \otimes_R H \longrightarrow H \otimes_R S \otimes_R H \otimes_R S$ given by $a \otimes b \longmapsto a \otimes 1 \otimes b$.

We denote by $\mathsf{Coalg}$ the category of coalgebras: its objects are pairs $(R \backslash H)$, where $R$ is an associative ring and $H = (H, \delta)$ is a coalgebra in the category of $R$-bimodules. Morphisms from $(R \backslash H)$ to $(R' \backslash H')$ are pairs $(\phi, \lambda_\phi)$, where $\phi$ is a ring morphism $R \longrightarrow R'$, $\lambda_\phi$ is a morphism of coalgebras $H_\phi \longrightarrow H'$ in the category of coalgebras in $R$-bimodules. The composition is defined in an obvious way.

We denote by $\mathsf{Coalg}_{fl}$ the full subcategory of $\mathsf{Coalg}$ whose objects are $(R \backslash H)$, $H = (H, \delta)$, such that $H$ is flat as a right $R$-module.

There is a forgetful functor $\Phi^* : \mathsf{Coalg} \longrightarrow \mathsf{Rings}$, $(R \backslash H) \longmapsto R$, $(\phi, \lambda_\phi) \longmapsto (\phi)$ (1) which is right adjoint to the functor

$$\Phi_* : \mathsf{Rings} \longrightarrow \mathsf{Coalg}, \quad R \longmapsto (R \backslash R), \quad \psi \longmapsto (\psi, \psi).$$

(2)

The adjunction morphisms are $\epsilon_\Phi = id : \Phi^* \Phi_* \longrightarrow Id_{\mathsf{Rings}}$ and $\eta_\Phi : Id_{\mathsf{Coalg}} \longrightarrow \Phi_* \Phi^*$; the latter morphism assigns to each object $(R \backslash H)$ of $\mathsf{Coalg}$ the morphism $(id_R, \epsilon_H)$, where $\epsilon_H$ denotes the counit $H \longrightarrow R$ of the coalgebra $H$ (which is a coalgebra morphism). Since the first adjunction arrow is an isomorphism, the functor $\Phi_*$ is fully faithful. In other words, the pair of functors $\mathsf{Coalg} \rightleftarrows \mathsf{Rings}$ is a $\mathsf{Q}^\circ$-category.

C6.2. The category $\mathfrak{Acov}$ of affine covers. On the other hand, consider the category $\mathfrak{Acov}$ of affine covers whose objects are flat, conservative, affine morphisms $\text{Sp}(R) \longrightarrow X$. Morphisms from $\text{Sp}(R) \xrightarrow{\phi} X$ to $\text{Sp}(S) \xrightarrow{\psi} Y$ are commutative diagrams

$$\begin{array}{ccc}
\text{Sp}(R) & \xrightarrow{f} & \text{Sp}(S) \\
\phi \downarrow & & \downarrow \psi \\
X & \xrightarrow{h} & Y
\end{array}$$

(3)

such that the morphism $\text{Sp}(R) \xrightarrow{f} \text{Sp}(S)$ is affine.

For each object $(R \backslash H)$ of the category $\mathsf{Coalg}$, the forgetful functor $(R \backslash H) - \text{Comod} \longrightarrow R - \mathsf{mod}$ is an inverse image functor of an affine morphism $\text{Sp}(R) \xrightarrow{f} \text{Sp}(R \backslash H)$. The morphism $f$ is flat iff $H$ is flat as a right $R$-module (see 7.3.1). Any morphism $(\phi, \lambda_\phi) : (R \backslash H) \longrightarrow (S \backslash G)$ induces a functor

$$|\phi, \lambda_\phi|^* : (R \backslash H) - \text{Comod} \longrightarrow (S \backslash G) - \text{Comod},$$

(4)
where \( \phi^*(M) = S \otimes_R M \), and the morphism \( \phi^*(M) \to G \otimes_S \phi^*(M) \) is the composition of

\[
\phi^*(\xi) : \phi^*(M) \to \phi^*(H \otimes_R M) = S \otimes_R H \otimes_R M,
\]

\[
S \otimes_R H \otimes_R M \to S \otimes_R H \otimes_R S \otimes_R M,
\]

and

\[
\lambda_{\phi} \otimes S M : S \otimes_R H \otimes_R S \otimes_R M \to G \otimes_S \phi^*(M) = G \otimes_R M.
\]

The functor \(|\phi, \lambda_\phi|^*\) is (regarded as) an inverse image functor of a morphism \(|\phi, \lambda_\phi| : \text{Sp}(S\backslash G) \to \text{Sp}(R\backslash H)\). The map \((\phi, \lambda_\phi) \mapsto |\phi, \lambda_\phi|^*\) extends to a pseudo-functor \(\text{Coalg} \to \text{Cat}\) which induces a functor \(\text{Coalg}^{op} \to |\text{Cat}|^{o}\). It follows that the diagram

\[
\begin{array}{ccc}
R \text{-} \text{mod} & \xrightarrow{\phi^*} & S \text{-} \text{mod} \\
\uparrow & & \uparrow \\
(R\backslash H) \text{-} \text{Comod} & \xrightarrow{|\phi, \lambda_\phi|^*} & (S\backslash G) \text{-} \text{Comod}
\end{array}
\]

(5)

commutes. In particular, the corresponding diagram in \(|\text{Cat}|^{o}\),

\[
\begin{array}{ccc}
\text{Sp}(S) & \xrightarrow{|\phi|} & \text{Sp}(R) \\
\downarrow & & \downarrow \\
\text{Sp}(S\backslash G) & \xrightarrow{|\phi, \lambda_\phi|} & \text{Sp}(R\backslash H)
\end{array}
\]

(6)

commutes. Thus we have a contravariant functor

\(\text{Coalg}^{op} \to \text{ACov}\)

(7)

from the category of coalgebras to the category of affine covers.

**C6.3. Proposition.** The image of the functor (7) is equivalent to the category of affine covers.

**Proof.** Let

\[
\begin{array}{ccc}
\text{Sp}(S) & \xrightarrow{f} & \text{Sp}(R) \\
\phi \downarrow & & \downarrow \psi \\
X & \xrightarrow{h} & Y
\end{array}
\]

(8)

be a morphism of covers, i.e. the morphisms \(\phi, \psi, f\) are affine and, in addition, \(\phi\) and \(\psi\) are flat and conservative.
Choosing inverse image functors of each of the morphisms in the diagram, we obtain a quasi-commutative diagram

\[
\begin{array}{ccc}
R \mod & \xrightarrow{f^*} & S \mod \\
\phi^* \uparrow & & \uparrow \psi^* \\
C_Y & \xrightarrow{h^*} & C_X
\end{array}
\] (9)

(a) Since \( f \) is affine, its direct image functor, \( f_* \), is the composition of an equivalence of the categories (Morita equivalence) \( S \mod \rightarrow S' \mod \) and the pull-back functor \( S' \mod \rightarrow R \mod \) by a ring morphism \( f : R \rightarrow S' \) (cf. 6.6.1). Thus the inverse image functor \( f^* \) of \( f \) is the composition of \( g \in S' \otimes_R - \rightarrow S - \mod \) and an equivalence \( S' \mod \rightarrow S - \mod \). Replacing \( S \) by \( S' \), we shall assume that the functor \( f^* \) in (9) is \( S \otimes_R - \rightarrow S \).

(b) The inverse image functor \( h^* \) defines a canonical functor morphism

\[ (\psi f)^*(\psi f)_* \xrightarrow{\lambda_h} \phi^* \phi_* \] (10)

Denote for convenience the composition \( \psi f \) by \( g \). The quasi-commutativity of (9) means that there is an isomorphism \( c = c_{\phi,h} : \phi^* h^* \sim g^* \). This isomorphism induces a morphism \( h^* \xrightarrow{\phi \phi_*} \phi_* g^* \). We define \( \lambda_h \) as the composition of the following morphisms:

\[ g^* g_* \xrightarrow{\phi \phi_*} \phi^* h^* g_* \xrightarrow{\phi \phi_*} \phi^* h^* g_* \xrightarrow{\phi \phi_*} \phi^* \phi_* \] (11)

(c) The morphism \( g^* g_* \xrightarrow{\phi \phi_*} \phi^* \phi_* \) is a comonad morphism.

The proof of this fact follows from the argument of Proposition 1.0.7.1 in [R4], where the similar fact is proven for the dual situation. One needs only to switch to dual categories and functors.

(d) Since the functors \( \phi^* \) and \( \psi^* \) are flat and conservative, the diagram (9) can be included into the diagram

\[
\begin{array}{ccc}
R \mod & \xrightarrow{f^*} & S \mod \\
\phi^* \uparrow & & \uparrow \psi^* \\
C_Y & \xrightarrow{h^*} & C_X
\end{array}
\]

\[
\begin{array}{ccc}
(R \backslash G_\phi) - \Comod & \xrightarrow{b^*} & (S \backslash G_\psi) - \Comod
\end{array}
\] (12)

Here \( G_\phi \) is the comonad \( (\phi^* \phi_*, \delta_\phi) \) and \( G_\psi = (\psi^* \psi_*, \delta_\psi) \), lower vertical arrows are canonical category equivalences given by Beck’s theorem. The functor \( b^* \) is induced by the monad morphism \( \lambda_h \) defined by (11) as follows. To any \( G_\psi \)-comodule \( L = (L, L \xrightarrow{\xi} \psi^* \psi_*(L)) \),
the functor $h^*$ assigns the $\mathcal{G}_\phi$-comodule $(f^*(L), \xi')$, where the $\mathcal{G}_\phi$-coalgebra structure, $\xi': f^*(L) \to \phi^*\phi_*f^*(L)$, is the composition of

\[ f^*(L) \xrightarrow{f^*(\xi)} f^*\psi^*\phi_*(L) \xrightarrow{f^*\psi^*\psi_*\eta_f} f^*\psi^*\psi_*f_*f^*(L) \xrightarrow{g^*g_*f^*(L)} \phi^*\phi_*f^*(L). \]

The diagram

\[ R - \text{mod} \xrightarrow{f^*} S - \text{mod} \]

\[ \phi_1^* \uparrow \]

\[ (R\backslash \mathcal{G}_\phi) - \text{Comod} \xrightarrow{h^*} (S\backslash \mathcal{G}_\psi) - \text{Comod} \]

obtained from (12) by composing the vertical arrows commutes. Here $\psi_1^*$ and $\phi_1^*$ are functors forgetting the respective comodule structures.

(c) It follows from (the argument of) 7.3.1 that the comonad $\mathcal{G}_\phi$ is isomorphic to the comonad $\mathcal{H}_\phi \otimes_R -$ for a coalgebra $\mathcal{H}_\phi = (H_\phi, \delta_\phi)$ in the category of $R$-bimodules determined uniquely up to isomorphism (see 7.3.1(2)). Therefore the category $(R\backslash \mathcal{G}_\phi) - \text{Comod}$ is naturally isomorphic to the category $(R\backslash \mathcal{H}_\phi) - \text{Comod}$ of $(R\backslash \mathcal{H}_\phi)$-comodules. Similarly, $(S\backslash \mathcal{G}_\psi) - \text{Comod} \simeq (S\backslash \mathcal{H}_\psi) - \text{Comod}$ for a coalgebra $\mathcal{H}_\psi$ in the category of $S$-bimodules.

The morphism $\lambda_\psi^*$ (defined by (11)) together with the ring morphism $R \xrightarrow{f} S$ (cf. (a)) define a coalgebra morphism $(\hat{f}, \lambda_\psi^*) : \mathcal{H}_\phi \to \mathcal{H}_\psi$ such that the diagram

\[ (R\backslash \mathcal{G}_\phi) - \text{Comod} \xrightarrow{h^*} (S\backslash \mathcal{G}_\psi) - \text{Comod} \]

\[ \uparrow \]

\[ (R\backslash \mathcal{H}_\phi) - \text{Comod} \xrightarrow{(\hat{f}, \lambda_\psi^*)} (S\backslash \mathcal{H}_\psi) - \text{Comod} \]

commutes. Combining (13) and (14), we obtain a commutative diagram

\[ R - \text{mod} \xrightarrow{f^*} S - \text{mod} \]

\[ \phi_1^* \uparrow \]

\[ (R\backslash \mathcal{H}_\phi) - \text{Comod} \xrightarrow{(\hat{f}, \lambda_\psi^*)} (S\backslash \mathcal{H}_\psi) - \text{Comod} \]

in which the vertical arrows are functors forgetting the comodule structure. This implies the assertion.

C6.4. Corollary. Let

\[ \begin{array}{ccc}
\text{Sp}(R) & \xrightarrow{f} & \text{Sp}(S) \\
\phi \downarrow & & \downarrow \psi \\
X & \xrightarrow{h} & Y 
\end{array} \]

be a morphism of affine flat covers. Then the morphism $X \xrightarrow{h} Y$ has a direct image functor.
Proof. Thanks to C6.3, it suffices to show that for any coalgebra morphism 

\[(\phi, \lambda_{\phi}) : (R\\mathcal{H}) \rightarrow (S\mathcal{G})\]

the functor 

\[|\phi, \lambda_{\phi}|^* : (R\mathcal{H}) – \text{Comod} \rightarrow (S\mathcal{G}) – \text{Comod}\]

has a right adjoint. This follows from 5.3.2.2. \(\blacksquare\)

C6.5. Proposition. Let \(\text{Sp}(R) \xrightarrow{\phi} X\) and \(\text{Sp}(S) \xrightarrow{\psi} Y\) be affine flat covers. Then \(X\) is isomorphic to \(Y\) iff there exist affine flat covers \(\text{Sp}(R) \xleftarrow{f} \text{Sp}(T) \xrightarrow{g} \text{Sp}(S)\) such that the associated coalgebras \(H_{\phi f}\) and \(H_{\psi g}\) in the category of \(T\)-bimodules are isomorphic.

Proof. (a) For any two affine flat covers, \(\text{Sp}(R) \xrightarrow{\phi} X\) and \(\text{Sp}(T) \xleftarrow{f} \text{Sp}(R)\), the categories \(H_{\phi} – \text{Comod}\) and \(H_{\phi f} – \text{Comod}\) are equivalent, or, what is the same, \(\text{Sp}(R\\mathcal{H}_{\phi}) \simeq \text{Sp}(T, H_{\phi f})\).

In fact, the composition of affine flat covers is an affine flat cover. Therefore, by Beck’s theorem, the category \(C_X\) is equivalent both to \(H_{\phi} – \text{Comod}\) and \(H_{\phi f} – \text{Comod}\).

(b) Let \(X \xleftarrow{\phi} \text{Sp}(R) \xleftarrow{f} \text{Sp}(T) \xrightarrow{g} \text{Sp}(S) \xrightarrow{\psi} Y\) be affine flat covers such that the coalgebras \(H_{\phi f}\) and \(H_{\psi g}\) are isomorphic. By (a), the category \(C_X\) is equivalent to \(H_{\phi f} – \text{Comod}\). Therefore, \(C_X\) is a Grothendieck category. In particular, it has arbitrary (small) colimits.

(c) Let \(\text{Sp}(R) \xrightarrow{\phi} X\) and \(\text{Sp}(S) \xrightarrow{\psi} Y\) be affine covers and \(\alpha\) an isomorphism \(Y \xrightarrow{\sim} X\). We claim that there exist affine covers \(\text{Sp}(R) \xleftarrow{f} \text{Sp}(T) \xrightarrow{g} \text{Sp}(S)\) such that the diagram

\[
\begin{array}{ccc}
\text{Sp}(T) & \xrightarrow{g} & \text{Sp}(S) \\
\phi \downarrow & & \psi \downarrow \\
\text{Sp}(R) & \xleftarrow{f} & \text{Sp}(T) \\
\phi & \downarrow & \psi \\
X & \xleftarrow{\alpha} & Y
\end{array}
\]

commutes. Replacing \(\psi\) by \(\alpha\psi\), we assume that \(\alpha\) is the identical isomorphism \(X \rightarrow X\).

The category \(C_X\) is equivalent to the category \((R\\mathcal{H}_{\phi}) – \text{Comod}\) for a coalgebra \(H_{\phi} = (H_{\phi}, \delta_{\phi})\) in the category of \(R\)-bimodules such that \(H_{\phi}\) is flat as a right \(R\)-module. This implies that \(C_X\) is a Grothendieck category. In particular, it has arbitrary (small) colimits.

Let \(\mathcal{F}_{\phi} = (F_{\phi}, \mu_{\phi}) = (\phi_*\phi^*, \mu_{\phi})\) and \(\mathcal{F}_{\psi} = (F_{\psi}, \mu_{\psi}) = (\psi_*\psi^*, \mu_{\psi})\) be monads associated with resp. \(\phi\) and \(\psi\). By 2.6.2.3 in [R4], there exists a free product, \(\mathcal{F}_{\phi} \star \mathcal{F}_{\psi}\), of the monads \(\mathcal{F}_{\phi}\) and \(\mathcal{F}_{\psi}\). The canonical monad morphisms \(\mathcal{F}_{\phi} \rightarrow \mathcal{F}_{\phi} \star \mathcal{F}_{\psi}\) induce a commutative diagram

\[
\begin{array}{ccc}
(\mathcal{F}_{\phi} \star \mathcal{F}_{\psi}/X) – \text{mod} & \rightarrow & (\mathcal{F}_{\phi}/X) – \text{mod} \\
\downarrow & & \downarrow \\
(\mathcal{F}_{\psi}/X) – \text{mod} & \rightarrow & C_X
\end{array}
\]

82
of the corresponding direct image functors. Since the monads \( \mathcal{F}_\phi \) and \( \mathcal{F}_\psi \) are continuous, their coproduct, \( \mathcal{F}_\phi \star \mathcal{F}_\psi \), is continuous too (by 2.6.2.3 in [R4]). This implies that all morphisms in the diagram (18) are affine. By Beck’s theorem, the category \( (\mathcal{F}_\phi/X) - \text{mod} \) is equivalent to the category \( R - \text{mod} \) and \( (\mathcal{F}_\psi/X) - \text{mod} \) is equivalent to the category \( S - \text{mod} \). Applying the Beck’s theorem to the (direct image of the) affine morphism \( (\mathcal{F}_\phi \star \mathcal{F}_\psi/X) - \text{mod} \to R - \text{mod} \), we obtain the commutative diagram

\[
\begin{array}{ccc}
T - \text{mod} & \longrightarrow & (\mathcal{F}_\phi \star \mathcal{F}_\psi/X) - \text{mod} \\
\downarrow f_* & & \downarrow \\
R - \text{mod} & \longrightarrow & (\mathcal{F}_\phi/X) - \text{mod}
\end{array}
\]

(19)

of (direct image) functors in which the horizontal arrows are category equivalences and the ring, the morphism \( f_* \) is the pull-back of a ring morphism \( f : R \to T \) which is defined uniquely up to isomorphism. Combining (18) and (19) and using an equivalence of \( (\mathcal{F}_\psi/X) - \text{mod} \) to \( S - \text{mod} \), we obtain a quasi-commutative diagram

\[
\begin{array}{ccc}
T - \text{mod} & \longrightarrow & S - \text{mod} \\
\downarrow f_* & \uparrow g_* & \downarrow \psi_* \\
R - \text{mod} & \longrightarrow & C_X
\end{array}
\]

(20)

Since the inverse image functors \( (\mathcal{F}_\phi/X) - \text{mod} \leftarrow C_X \longrightarrow (\mathcal{F}_\psi/X) - \text{mod} \) are flat and conservative, the inverse image functors

\[
(\mathcal{F}_\phi/X) - \text{mod} \longrightarrow (\mathcal{F}_\phi \star \mathcal{F}_\psi/X) - \text{mod} \leftarrow (\mathcal{F}_\psi/X) - \text{mod}
\]

have the same property (by 2.6.4.1, or 4.3.2 in [R4]). Therefore \( f_* \) and \( g_* \) in (19) are direct image functors of affine flat covers.

C6.5.1. The dual description. Let \( \text{Sp}(R) \xrightarrow{f} X \) be an affine morphism. Denote by \( f^\wedge \) the dual morphism \( X \longrightarrow \text{Sp}(R) \) with an inverse image functor \( f_* \) (cf. 3.4). Since \( f \) is affine, the morphism \( f^\wedge \) is continuous: \( f_*^\wedge = f^! \). By 4.1, the morphism \( f^\wedge \) (hence the morphism \( f \)) is uniquely defined by the right \( R \)-module \( (\mathcal{O}_*, \hat{\phi}) \), where \( \mathcal{O}_* = f^\wedge(\mathcal{O}) = f_*^!(\mathcal{O}) \) and \( \hat{\phi} \) is the canonical ring morphism \( R \longrightarrow C_X(\mathcal{O}_*, \mathcal{O}_*)^\circ = C_X(f_*^!(\mathcal{O}), f_*^!(\mathcal{O}))^\circ \). The functor \( f_*^\wedge = f^! \) is the composition of the functor

\[
C_X(\mathcal{O}_*, -) : C_X \longrightarrow C_X(\mathcal{O}_*, \mathcal{O}_*)^\circ - \text{mod}
\]

and the pull-back

\[
\hat{\phi}_* : C_X(\mathcal{O}_*, \mathcal{O}_*)^\circ - \text{mod} \longrightarrow R - \text{mod}
\]

by the ring morphism \( \hat{\phi} \). Let \( X = \text{Sp}(R\backslash \mathcal{H}) \), where \( \mathcal{H} = (H, \delta) \) is a coalgebra in the category of \( R \)-bimodules, and let \( f \) be the standard morphism \( \text{Sp}(R) \longrightarrow \text{Sp}(R\backslash \mathcal{H}) \) having as an inverse image functor the forgetful functor \( (R\backslash \mathcal{H}) - \text{Comod} \longrightarrow R - \text{mod} \) (cf. 7.3.1(1))

83
and as a direct image functor the functor $L \mapsto \mathcal{H} \otimes_R L$ (cf. 7.3.1(2)). Then the object $\mathcal{O}_*$ coincides with the $\mathcal{H}$-comodule $\mathcal{H} = (H, \delta)$, and the functor $\tilde{f}_* = f^!$ is isomorphic to

$$(R \setminus \mathcal{H}) - \text{Comod} \longrightarrow R - \text{mod}, \quad L \mapsto \text{Hom}_R(H, L) \quad (1)$$

**C6.6. Affine covers over the spectrum of a ring.**

**C6.6.1. Affine covers over $\text{Sp}(S)$.** Fix an associative ring $S$ and denote by $\text{ACov}/\text{Sp}(S)$ the category of affine covers over $\text{Sp}(S)$. Its objects are diagrams

$$\text{Sp}(R) \xrightarrow{\phi} X \xrightarrow{f} \text{Sp}(S)$$

such that $\phi$ is affine, flat, and conservative, and the composition $f \circ \phi$ is affine. Morphisms from $\text{Sp}(R) \xrightarrow{\phi} X \xrightarrow{f} \text{Sp}(S)$ to $\text{Sp}(T) \xrightarrow{\psi} Y \xrightarrow{g} \text{Sp}(S)$ are commutative diagrams

$$\begin{array}{ccc}
\text{Sp}(R) & \xrightarrow{\gamma} & \text{Sp}(T) \\
\phi \downarrow & & \downarrow \psi \\
X & \xrightarrow{h} & Y \\
\text{Sp}(S) & \xleftarrow{g} & \text{Sp}(S)
\end{array} \quad (1)$$

The composition is defined in an obvious way. Notice that, since the compositions $f \circ \phi$ and $g \circ \psi$ in (1) are affine, the morphism $\gamma$ is affine too (see 6.4).

**C6.6.2. Coalgebras over a ring.** For convenience, we denote by $S \setminus \text{Coalg}$ the category $(S \setminus S) \setminus \text{Coalg}$ of coalgebras over the coalgebra $(S \setminus S)$. It follows from definitions that the category $S \setminus \text{Coalg}$ admits the following description. Objects of $S \setminus \text{Coalg}$ are triples $((R \setminus \mathcal{H}), \phi, \tau)$, where $(R \setminus \mathcal{H})$ is an object of the category $\mathcal{Coalg}$ (i.e. $\mathcal{H} = (H, \delta)$ is a coalgebra in the category of $R$-bimodules), $\phi$ is a ring morphism $S \longrightarrow R$, and $\tau$ is a coalgebra morphism $(R \setminus R \otimes_S R) \longrightarrow (R \setminus \mathcal{H})$. In particular, $\epsilon \circ \tau : R \otimes_S R \longrightarrow R$ is the morphism induced by the multiplication, $\mu_R$, on $R$ (cf. C6.1). Here $\epsilon$ denotes the counit, $H \longrightarrow R$, of the coalgebra $\mathcal{H}$.

For any ring $S$, the map which assigns to every object $((R \setminus \mathcal{H}), S \xrightarrow{\phi} R, \tau,)$ of the category $S \setminus \text{Coalg}$ the diagram

$$\text{Sp}(R) = \text{Sp}^o(R \setminus R) \xrightarrow{\text{Sp}^o(\epsilon)} \text{Sp}^o(R \setminus \mathcal{H}) \xrightarrow{\text{Sp}^o(\phi, \tau)} \text{Sp}^o(S \setminus S) = \text{Sp}(S) \quad (2)$$

extends naturally to a functor

$$(S \setminus \text{Coalg})^{op} \longrightarrow \text{ACov}/\text{Sp}(S) \quad (3)$$

**C6.6.3. Proposition.** The image of the functor (3) is equivalent to the category $\text{ACov}/\text{Sp}(S)$. 

84
Proof. Let \( \text{Sp}(R) \xrightarrow{\phi} X \xrightarrow{f} \text{Sp}(S) \) be an affine cover over \( \text{Sp}(S) \). Replacing it by an isomorphic cover, we can assume that \( X = \text{Sp}^\circ(R\backslash H) \) for a coalgebra \( (R\backslash H) \), the inverse image \( \phi^* \) of \( \phi \) is the forgetful functor \( (R\backslash H) - \text{Comod} \to R - \text{mod} \), and the composition \( f \circ \phi \) corresponds to a ring morphism \( S \to R \). The latter means that \((f \circ \phi)^*(S) \cong \phi^*(f^*(S)) \cong R \), where the isomorphisms are \((R, S)\)-bimodule isomorphisms. We can assume that \( \phi^*(f^*(S)) = R \). Since \( \phi^* \) is the functor forgetting the coaction, the \((R\backslash H)\)-comodule \( f^*(S) \) is of the form \((R, \gamma)\) for some coaction \( \gamma : R \to H \otimes_R R \cong H \). The fact that the \( R \)-module morphism \( \gamma \) is an \( H \)-comodule structure and an \((R, S)\)-bimodule morphism implies that its composition with the natural isomorphism \( H \otimes R R \to H \) induces a coalgebra morphism \((R\backslash R \otimes_S R) \to (R\backslash H) \). Thus we have assigned to an affine cover over \( \text{Sp}(S) \) a coalgebra over \( S \) which the functor (3) maps to an affine cover that is isomorphic to the affine cover we have started with. \( \square \)

C6.6.4. Remark. Proposition C6.6.3 can be also deduced directly from C6.3 as follows. Since \( Z \) is the initial object of the category \( \text{Rings} \) of unital rings, the fully faithful functor

\[
\Phi_* : \text{Rings} \longrightarrow \text{Coalg}, \quad R \mapsto (R\backslash R),
\]

induces a fully faithful functor \( \Phi^+_* : \text{Rings} \to \mathbb{Z}\text{Coalg} \) which is right adjoint to the forgetful functor

\[
\Phi^{+*} : \mathbb{Z}\text{Coalg} \longrightarrow \text{Rings}, \quad (R\backslash H, R \otimes_Z R \xrightarrow{\tau} H) \mapsto R.
\]

(cf. C6.1). Moreover, for any ring \( S \) the natural functor

\[
(S\backslash S)\backslash (\mathbb{Z}\text{Coalg}) \longrightarrow (S\backslash S)\backslash \text{Coalg}
\]
is an isomorphism of categories.

C7. Affine flat covers and descent of flat morphisms. If \( U = \text{Sp}(R) \) and \( V = \text{Sp}(S) \), then the category \( C_{\text{Hom}^c(U, V)} \) is equivalent to the category of \((R, S)\)-bimodules, or, equivalently, \( \text{Hom}^c(U, V) \) is isomorphic to \( \text{Sp}(R \otimes S^\circ) \) (cf. 9.2.1).

The object \( \text{Hom}_w^f(U, V) \) corresponds to the full subcategory of the category of \((R, S)\)-bimodules whose objects are bimodules which are flat as right \( S \)-modules. We shall write \( \text{Hom}_f(U, V) \) instead of \( \text{Hom}_w^f(U, V) \).

Notice that since the categories \( C_X \) and \( C_Y \) are abelian, weakly flat morphisms are just flat. By this reason, we shall write \( \text{Hom}_f(X, Y) \) instead of \( \text{Hom}_w^f(X, Y) \).

Let \( U \xrightarrow{u} X \) and \( V \xrightarrow{v} Y \) be affine flat covers (i.e. affine, flat, conservative morphisms); hence \( X \simeq \text{Sp}^\circ(R\backslash H_u) \) and \( Y \simeq \text{Sp}^\circ(S\backslash H_v) \), where \( \mathcal{H} = (H_u, \delta_u) \) is a coalgebra in the category of \( R \)-bimodules and \( \mathcal{H}_v = (H_v, \delta_v) \) is a coalgebra in the category of \( S \)-bimodules. The coalgebras \( \mathcal{H}_u, \mathcal{H}_v \) determine a coalgebra \( \mathcal{H}_{(u|v)} = (H_{(u|v)}, \delta_{(u|v)}) \) in the category of \( R \otimes S^\circ \)-bimodules which is naturally identified with \( \mathcal{H}_u \otimes \mathcal{H}_v^\circ \). The latter follows from the fact that the tensoring by \( H_{(u|v)} \) maps the bimodule \( R \otimes S^\circ \) to \( H_u \otimes H_v \) (i.e. \( H_{(u|v)} = H_u \otimes H_v \)), and the comultiplication \( \delta_{(u|v)} \) on \( H_u \otimes H_v \) is the one induced by the comultiplications \( (\delta_u, \delta_v) \).

C7.1. Proposition. Let \( \text{Sp}(R) \xrightarrow{u} X \) and \( \text{Sp}(S) \xrightarrow{v} Y \) be affine flat covers. Then
(a) The pair \((u, v)\) determines an affine morphism

\[
\text{Sp}(R \otimes S^o) \longrightarrow \text{Hom}^c(X, Y).
\]

which is the decomposition of

\[
\text{Sp}(R \otimes S^o) \longrightarrow \text{Sp}^o(R \otimes S^o \backslash \mathcal{H}_u \otimes \mathcal{H}_v^o)
\]

and

\[
\text{Sp}^o(R \otimes S^o \backslash \mathcal{H}_u \otimes \mathcal{H}_v^o) \longrightarrow \text{Hom}^c(X, Y).
\]

(b) The morphism (3) is an exact localization (i.e. its inverse image functor is an exact localization). Equivalently, an inverse image functor of (3) is exact and its direct image functor is fully faithful.

(c) The morphism (1) (or (3)) induces an equivalence of the category \(C_{\text{Hom}^c(X, Y)}\) of flat morphisms \(X \longrightarrow Y\) and the full subcategory of the category \((R \otimes S^o \backslash \mathcal{H}_u \otimes \mathcal{H}_v^o)\text{-Comod}\) whose objects are comodules \((M, \xi)\) such that the \(R \otimes S^o\)-module \(M\) is flat as a right \(S\)-module.

Proof. The assertion follows from 9.7 and the Beck’s theorem 5.4.1. ■

C7.2. Morphisms corresponding to bicomodules. Under the conditions of C7.1, we can identify \(X\) with \(\text{Sp}^o(R \backslash \mathcal{H}_u)\) and \(Y\) with \(\text{Sp}^o(S \backslash \mathcal{H}_v)\).

One can define a (fully faithful) direct image functor of the morphism (11) as follows. Let \(M\) be a \(H_u \otimes S\)-comodule which is convenient to regard as an \((H_u, H_v)\)-bicomodule, \(M = (M, \zeta_u, M \otimes S \mathcal{H}_v).\) It follows from 10.7.1 that a direct image functor of the morphism (3) assigns to \(M\) a functor

\[
C_Y = (S \backslash \mathcal{H}_v) \rightarrow \text{Comod} \longrightarrow (R \backslash \mathcal{H}_u) \rightarrow \text{Comod} = C_X
\]

which maps each \(\mathcal{H}_v\)-comodule, \(L = (L, \xi_L)\) to a kernel of the pair of morphisms

\[
M \otimes S L \xrightarrow{M \otimes S \xi_L} M \otimes_S H_v \otimes S L.
\]

with a comodule structure induced by the coaction \(\zeta_u : M \longrightarrow H_u \otimes_S M.\)

C7.2.1. Composition of morphisms. Let \(u : \text{Sp}(R) \longrightarrow X, \ v : \text{Sp}(S) \longrightarrow Y\) and \(v : \text{Sp}(T) \longrightarrow Z\) be affine flat morphisms which allows to assume that \(X = \text{Sp}^o(R \backslash \mathcal{H}_u),\ Y = \text{Sp}^o(S \backslash \mathcal{H}_v),\) and \(Z = \text{Sp}^o(T \backslash \mathcal{H}_w)\). Let \(M = (M, \zeta_u, \zeta_v)\) be a \(((R \backslash \mathcal{H}_u), (S \backslash \mathcal{H}_v))\)-bicomodule and \(N = (N, \zeta_w, \zeta_v)\) a \(((S \backslash \mathcal{H}_v), (T \backslash \mathcal{H}_w))\)-bicomodule.

Denote by \(M \boxtimes_{(S \backslash \mathcal{H}_v)} N,\) or simply by \(M \boxtimes N,\) the kernel of the diagram

\[
M \otimes_S N \xrightarrow{\zeta_v} M \otimes_S H_v \otimes_S N.
\]
The right coaction $\zeta^w : N \longrightarrow N \otimes_S H_w$ induces a right $(T \setminus \mathcal{H}_w)$-comodule structure, and the left $(R \setminus \mathcal{H}_u)$-comodule structure $\zeta_u : M \longrightarrow H_u \otimes_S M$ induces a left $(R \setminus \mathcal{H}_u)$-comodule structure on $M \boxtimes N$. Thus we obtain a $((R \setminus \mathcal{H}_u), (T \setminus \mathcal{H}_w))$-bicomodule $M \boxtimes N = (M \boxtimes N, \zeta_u \boxtimes N, M \boxtimes \zeta^w)$.

There is a natural morphism from the composition $\phi_M^* \circ \phi_N^*$ of the functors corresponding to the resp. bicomodules $N$ and $M$ to the functor

$$\phi_{M \boxtimes N}^* : (T \setminus \mathcal{H}_w) - \text{Comod} \longrightarrow (R \setminus \mathcal{H}_u) - \text{Comod}$$

corresponding to the $(R \setminus \mathcal{H}_u), (T \setminus \mathcal{H}_w))$-bicomodule $M \boxtimes N$. This morphism is an isomorphism if $M$ is flat as a right $S$-module.

### C7.3. The bicategory of affine flat covers.

We define the bicategory of affine flat covers as follows. Its objects are affine flat covers $\text{Sp}(R) \xrightarrow{u} X$. The category of morphisms from $\text{Sp}(R) \xrightarrow{u} X$ to $\text{Sp}(S) \xrightarrow{v} Y$ is the category of $((R \setminus \mathcal{H}_u), (S \setminus \mathcal{H}_v))$-bicomodules. The composition is given by the tensor product (\boxtimes, see C7.2.1) of bicomodules.

#### C7.3.1. Proposition. The category of morphisms from a flat cover $\text{Sp}(R) \xrightarrow{u} X$ to a flat cover $\text{Sp}(S) \xrightarrow{v} Y$ is determined by the objects $X$ and $Y$ uniquely up to equivalence.

**Proof.** Let $\text{Sp}(R) \xrightarrow{u} X$ and $\text{Sp}(R') \xrightarrow{u'} X$ be two affine flat covers of $X$. By C6.5, there exists affine flat covers $\text{Sp}(R) \xrightarrow{p} \text{Sp}(T) \xrightarrow{p'} \text{Sp}(R')$ such that $u \circ p = u' \circ p'$. Thus, it suffices to show that the category of morphisms from $\text{Sp}(R) \xrightarrow{u} X$ to $\text{Sp}(S) \xrightarrow{v} Y$ is equivalent to the category of morphisms from $\text{Sp}(T) \xrightarrow{u} X$ to $\text{Sp}(S) \xrightarrow{v} Y$.

The functor which assigns to every $((R \setminus \mathcal{H}_u), (S \setminus \mathcal{H}_v))$-bicomodule $\mathcal{M} = (M, \zeta_u, \zeta_v)$ the $((T \setminus \mathcal{H}_wp), (S \setminus \mathcal{H}_v))$-bicomodule $T \otimes_S \mathcal{M} = (M, \zeta_{wp}, \zeta_v)$ is a category equivalence. This follows from 7.4.5, or directly from the Beck's theorem (see the part (a) of the argument of 7.4.5). $\blacksquare$

### C7.4. Descent of morphisms over $\text{Sp}(T)$.

Fix an object $Z$ of the category $|\text{Cat}|^o$. For any two objects $X = (X, X \xrightarrow{f} Z)$, $Y = (Y, Y \xrightarrow{f} Z)$ of the category $|\text{Cat}|^o/Z$, we denote by $\text{Hom}_Z(X, Y)$ (resp. $\text{Hom}_Z^e(X, Y)$, resp. $\text{Hom}_Z^{fl}(X, Y)$) the object of $|\text{Cat}|^o$ corresponding to the category of functors $\phi^* : C_Y \longrightarrow C_X$ (resp. having a right adjoint, resp. exact and having a right adjoint) such that the diagram

$$
\begin{array}{ccc}
C_Y & \xrightarrow{\phi^*} & C_X \\
g^* & & f^* \\
\downarrow & & \uparrow \\
C_Z & \\
\end{array}
$$

quasi-commutes.

Suppose that $Z = \text{Sp}(T)$ for an associative unital ring $T$; and let $u : \text{Sp}(R) \longrightarrow X$ and $v : \text{Sp}(S) \longrightarrow Y$ be affine flat covers such that the compositions $\text{Sp}(R) \xrightarrow{fu} \text{Sp}(T) \xrightarrow{gv} \text{Sp}(S)$ are affine. Thanks to 7.6.3 (see also 7.6.2), we can assume that the compositions $f \circ u$, $g \circ v$ correspond to ring morphisms resp. $T \longrightarrow R$ and $T \longrightarrow S$, and $X = \text{Sp}^o(R \setminus \mathcal{H}_u)$...
and $Y = \text{Sp}^o(S \setminus H_v)$ for coalgebras $(R \setminus H_u)$ and $(S \setminus H_v)$ in resp. the category of $R$- and $S$-bimodules. The morphisms $X \rightarrow \text{Sp}(T)$ and $Y \rightarrow \text{Sp}(T)$ are described by resp.

$$(T \phi_f, R \rightarrow (R \setminus R \otimes_T R) \xrightarrow{\tau_u} (R \setminus H_u)) \quad \text{and} \quad (T \phi_f, S \rightarrow (S \setminus S \otimes_T S) \xrightarrow{\tau_u} (S \setminus H_v)),$$

(cf. C6.6.2). The quasi-commutativity of the diagram (5) implies in particular that $\phi^*(\mathcal{O}_Y) \simeq \mathcal{O}_X$. Here $\mathcal{O}_X = f^*(\mathcal{O}_T) = f^*(T)$, and $\mathcal{O}_Y = g^*(\mathcal{O}_T) = g^*(T)$. The morphisms $f$ and $g$ are continuous (this follows from 7.6.3). Therefore, by 3.4.1, the object $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$) determines the morphism $f$ (resp. $g$) uniquely up to isomorphism. This, in turn, implies that the diagram (5) quasi-commutes if and only if $\phi^*(\mathcal{O}_Y) \simeq \mathcal{O}_X$. The object $\mathcal{O}_X = f^*(\mathcal{O}_T) = f^*(T)$ is (isomorphic to) the $(R \setminus H_u)$-comodule $(R, R \xrightarrow{\tau'_u} H_u)$, and $\mathcal{O}_Y = g^*(\mathcal{O}_T) = g^*(T)$ is (isomorphic to) the $(S \setminus H_v)$-comodule $(S, S \xrightarrow{\tau'_v} H_v)$. The coactions $\tau_u$ and $\tau'_v$ induce the coalgebra morphisms $(R \setminus R \otimes_T S) \xrightarrow{\tau_v} (R \setminus H_u)$ and $(S \setminus S \otimes_T S) \xrightarrow{\tau_v} (S \setminus H_v)$ which appear above (cf. C6.6.2).

Suppose the functor $\phi^* : C_Y \rightarrow C_X$ is given by an $((R \setminus H_u), (S \setminus H_v))$-bicomodule $M = (M, H_u \otimes_R M \xrightarrow{\zeta_u} M \xrightarrow{\zeta_v} M \otimes_S H_v)$. Then by C7.2 applied to the $(S \setminus H_v)$-comodule $\mathcal{O}_Y = (S, \tau_v)$, the $(R \setminus H_u)$-comodule $\phi^*(\mathcal{O}_Y)$ is a kernel of the pair of morphisms

$$M \xrightarrow{M \otimes S \tau_v} M \xrightarrow{\zeta_v} M \otimes_S H_v. \tag{6}$$

with a comodule structure induced by the coaction $\zeta_u : M \rightarrow H_u \otimes_R M$.

Therefore, the diagram (5) is quasi-commutative iff the kernel of (6) is isomorphic to $u^*(\mathcal{O}_X) = R$ and the $(R \setminus H_u)$-comodule structure induced on $R$ by $M \xrightarrow{\zeta_u} H_u \otimes_R M$ is isomorphic to $\tau_u : R \rightarrow H_u$.

The argument above proves the following

**C7.4.1. Proposition.** An $((R \setminus H_u), (S \setminus H_v))$-bicomodule $M = (M, \zeta_u, \zeta_v)$ determines a continuous morphism from $X = (X, X \xrightarrow{f} \text{Sp}(T))$ to $Y = (Y, Y \xrightarrow{g} \text{Sp}(T))$ iff there exists an $(R \setminus H_u)$-comodule morphism $\lambda : (R, \tau_u) \rightarrow (M, \zeta_u)$ such that the diagram

$$R \xrightarrow{\lambda} M \xrightarrow{M \otimes S \tau_v} M \otimes_S H_v \tag{7}$$

is exact.

**C7.4.2. Corollary.** The category of flat morphisms from $X = (X, X \xrightarrow{f} \text{Sp}(T))$ to $Y = (Y, Y \xrightarrow{g} \text{Sp}(T))$ is equivalent to the category whose objects are pairs $(\mathcal{M}, \lambda)$, where $\mathcal{M} = (M, \zeta_u, \zeta_v)$ is an $((R \setminus H_u), (S \setminus H_v))$-bicomodule such that $M$ is flat as a right $S$-module, and $\lambda$ an $(R \setminus H_u)$-comodule morphism $(R, \tau_u) \rightarrow (M, \zeta_u)$ such that the diagram

$$R \xrightarrow{\lambda} M \xrightarrow{M \otimes S \tau_v} M \otimes_S H_v \tag{7}$$

is exact.
is exact. Morphisms \((\mathcal{M}, \lambda) \to (\mathcal{M}', \lambda')\) are \(((R\backslash\mathcal{H}u), (S\backslash\mathcal{H}u))\)-bicomodule morphisms \(\psi : \mathcal{M} \to \mathcal{M}'\) which make the diagram

\[
\begin{array}{ccc}
(M, \zeta_u) & \xrightarrow{\psi} & (M', \zeta'_u) \\
\downarrow{\lambda} & & \downarrow{\lambda'} \\
(R, \tau_u) & & 
\end{array}
\]

(8)

commute.

Proof. The fact follows from C7.1(c) and C7.4.1. Details are left to the reader. ■

C8. Flat descent and relations.

C8.1. Proposition. (a) Let \(\mathcal{X} \xrightarrow{P_1} \mathcal{Y} \xrightarrow{f} X\) be an exact diagram in \(|\text{Cat}|^o\). Then
(i) The morphism \(f\) (i.e. its inverse image functor \(f^\ast\)) is conservative.
(ii) If the functors \(C_{\mathcal{Y}} \xrightarrow{\nu_1^*} C_{\mathcal{X}}\) preserve limits (resp. colimits) of a certain type, then \(f^\ast\) has the same property.
(b) Let \(\mathcal{X} \xleftarrow{P} \mathcal{Z} \xrightarrow{q} \mathcal{Y}\) be a diagram in \(|\text{Cat}|^o\). Suppose the functors \(P^\ast\) and \(q^\ast\) preserve limits (resp. colimits) of a certain type, then the inverse image functors of the canonical coprojections

\[
\begin{array}{c}
\mathcal{X} \xrightarrow{\pi_P} \mathcal{X} \coprod_{P,q} \mathcal{Y} \xleftarrow{\pi_q} \mathcal{Y}
\end{array}
\]

have the same property.

Proof. (a) By 2.2, the category \(C_X\) is the kernel of the pair \(C_{\mathcal{Y}} \xrightarrow{\nu_1^*} C_{\mathcal{X}}\) of inverse image functors of \(p_1\) and \(p_2\). This means that the category \(C_X\) can be described as follows: its objects are pairs \((M, \phi)\), where \(M \in \text{Ob}C_{\mathcal{Y}}\) and \(\phi\) is an isomorphism \(p_1^\ast(M) \cong p_2^\ast(M)\). A morphism from \((M_1, \phi_1)\) to \((M_2, \phi_2)\) is given by a morphism \(M_1 \xrightarrow{g} M_2\) such that the diagram

\[
\begin{array}{ccc}
p_1^\ast(M_1) & \xrightarrow{p_1^\ast(g)} & p_1^\ast(M_1) \\
\phi_1 \downarrow & & \downarrow \phi_2 \\
p_2^\ast(M_1) & \xrightarrow{p_2^\ast(g)} & p_2^\ast(M_1)
\end{array}
\]

(1)

commutes. A natural inverse image functor, \(C_X \xrightarrow{f^\ast} C_{\mathcal{Y}}\), of the morphism \(f\) assigns to every object \((M, \phi)\) of \(C_X\) the object \(M\).

(i) It follows from the construction that \(f^\ast\) is conservative.
(ii) If \(p_1^\ast\), \(p_2^\ast\) preserve a certain type of limits (resp. colimits), it follows from the description of the category \(C_X\) that the functor \((M, \phi) \xrightarrow{f^\ast} M\) preserves the same type of limits (resp. colimits).
(b) The category $C_X \coprod_{p,q} Y$ is described as follows. Its objects are triples $(M, \phi, L)$, where $M \in \text{Ob}C_X$, $L \in \text{Ob}C_Y$, and $\phi$ is an isomorphism $p^*(M) \overset{\sim}{\to} q^*(L)$. Morphisms $(M, \phi, L) \to (M', \phi', L')$ are pairs of morphisms $(M \overset{u}{\to} M', L \overset{v}{\to} L')$ such that the diagram

$$
\begin{array}{ccc}
p^*(M) & \overset{\phi}{\to} & q^*(L) \\
p^*(u) \downarrow & & \downarrow q^*(v) \\
p^*(M') & \overset{\phi'}{\to} & q^*(L')
\end{array}
$$

commutes. Composition is defined in an obvious way. A natural inverse image functor of the projection $\pi_p$ (resp. $\pi_q$) maps an object $(M, \phi, L)$ to $M$ (resp. to $L$) and a morphism $(M, \phi, L) \xrightarrow{(u,v)} (M', \phi', L')$ to $M \overset{u}{\to} M'$ (resp. to $L \overset{v}{\to} L'$).

It follows from this description (as in (ii) above) that if the functors $p^*$ and $q^*$ preserve (co)limits of certain type, then the functors $\pi_p^*$ and $\pi_q^*$ have the same property.

**C8.1.1. Corollary.** Let $\mathfrak{F} = (\mathcal{F} \overset{\pi}{\to} \mathcal{E})$ be a fibered category.

(a) Let $\mathfrak{R} \xrightarrow{p_1} \mathfrak{G} \xrightarrow{f} \mathfrak{X}$ be an exact diagram of presheaves of sets on the category $\mathcal{E}$.

Then

(i) The morphism $f$ is conservative.

(ii) If the functors $\text{Qcoh}(\mathfrak{F}/\mathfrak{R}) \xrightarrow{p_1^*} \text{Qcoh}(\mathfrak{F}/\mathfrak{G})$ preserve a certain type of limits (resp. colimits), then $f^*$ has the same property.

(b) Let $X \xrightarrow{p} Z \xrightarrow{q} Y$ be a diagram of presheaves of sets on the category $\mathcal{E}$. If the functors $p^*$ and $q^*$ preserve limits (resp. colimits) of a certain type, then the inverse image functors of the canonical coprojections

$$
X \xrightarrow{\pi_p} X \coprod_{p,q} Y \xleftarrow{\pi_q} Y
$$

have the same property.

**Proof.** By [KR3, 11.1.5.2(b)], the category $\text{Qcoh}(\mathfrak{F}/\mathfrak{X})$ of quasi-coherent modules on $\mathfrak{X}$ is the kernel of the pair $\text{Qcoh}(\mathfrak{F}/\mathfrak{G}) \xrightarrow{p_1^*} \text{Qcoh}(\mathfrak{F}/\mathfrak{R})$ of inverse image functors of $p_1$ and $p_2$, and the category $\text{Qcoh}(\mathfrak{F}/X \coprod_{p,q} Y)$ is the pull-back of the inverse image functors $\text{Qcoh}(\mathfrak{F}/X) \xrightarrow{p^*} \text{Qcoh}(\mathfrak{F}/Z) \xleftarrow{q^*} \text{Qcoh}(\mathfrak{F}/Y)$. The assertion follows now from C8.1. ■

**C8.1.1.1. Proposition.** Let $\mathfrak{F} = (\mathcal{F} \overset{\pi}{\to} \mathcal{E})$ be a fibered category, and let $\mathfrak{S}$ be a topology on $\mathcal{E}$ which is coarser than the topology of effective descent.

(a) Let $\mathfrak{R} \xrightarrow{p_1} \mathfrak{G} \xrightarrow{f} X$ be an exact diagram of sheaves of sets on $(\mathcal{E}, \mathfrak{S})$. Then

(i) The morphism $f$ is conservative.

(ii) If the functors $\text{Qcoh}(\mathfrak{F}/\mathfrak{R}) \xrightarrow{p_1^*} \text{Qcoh}(\mathfrak{F}/\mathfrak{G})$ preserve a certain type of limits (resp. colimits), then $f^*$ has the same property.
(b) Let

\[ \begin{array}{ccc}
\mathfrak{T} & \xrightarrow{p} & \mathfrak{X} \\
\mathfrak{Y} & \xrightarrow{\pi_q} & \mathfrak{X} \coprod_{p,q} \mathfrak{Y}
\end{array} \]

be a cartesian coproduct of sheaves of sets on \((\mathcal{E}, \mathcal{T})\). If the functors \(p^*, q^*\) preserve limits (resp. colimits) of a certain type, then the functors \(\pi_{p}^*, \pi_{q}^*\) have the same property.

**Proof.** Let \(X'\) be the cokernel of the pair \(\mathfrak{R} \xrightarrow{\mathfrak{G}} \mathfrak{Y}\) in the category \(\mathcal{E}^\wedge\) of presheaves on \(\mathcal{E}\). Since the topology \(\mathcal{T}\) is coarser than the topology of effective descent and \(X\) is a sheaf associated with the presheaf \(X'\), the unique presheaf morphism \(X' \rightarrow X\) induces an equivalence of categories \(\text{Qcoh}(\mathfrak{F}/X) \xrightarrow{\sim} \text{Qcoh}(\mathfrak{F}/X')\) (see [KR3, 2.6]. The assertion follows now from C8.1.1.

C8.2. **Proposition.** Let \(\begin{array}{ccc}
\mathfrak{X} & \xrightarrow{p_1} & \mathfrak{Y} \\
\mathfrak{Y} & \xrightarrow{p_2} & \mathfrak{X}
\end{array} \xrightarrow{f} X\) be an exact diagram in \(|\text{Cat}|^o\) such that

(i) inverse image functors of \(p_1\) and \(p_2\) preserve kernels of coreflexive pairs of arrows;
(ii) the morphism \(f\) is continuous;
(iii) The category \(C_{\mathfrak{Y}}\) has kernels of coreflexive pairs of arrows.

Then the morphism \(f\) is comonadic, i.e. \(\mathfrak{Y} \xrightarrow{f} X\) is isomorphic to the canonical morphism \(\mathfrak{Y} \rightarrow \text{Sp}^o(\mathfrak{Y} \setminus \mathcal{H}_f)\), where \(\mathcal{H}_f\) is a comonad associated with (a choice of inverse and direct image functors and adjunction morphisms of) the morphism \(f\).

**Proof.** By C8.1, the morphism \(f\) is weakly flat, i.e. its inverse image functor preserves kernels of coreflexive pairs of arrows. The assertion now follows from the Beck’s theorem (see 5.4.1).

C8.2.1. **Remark.** Let \(\begin{array}{ccc}
\mathfrak{X} & \xrightarrow{p_1} & \mathfrak{Y} \\
\mathfrak{Y} & \xrightarrow{p_2} & \mathfrak{X}
\end{array} \xrightarrow{f} X\) be an exact diagram in \(|\text{Cat}|^o\). Suppose the morphisms \(p_1, p_2\) in the diagram are continuous. Then their inverse image functors resp. \(p_1^*\) and \(p_2^*\) preserve colimits of any small diagram. By C8.1(ii), this implies that \(f^*\) preserves colimits of any small diagram. Recall that if the category \(C_X\) has colimits of small diagrams, \(f\) is continuous iff the following conditions hold (cf. [BD]):

(a) \(f^*\) preserves colimits of small diagrams;
(b) for any \(W \in \text{Ob}C_{\mathfrak{Y}}\), there exists a morphism \(f^*(V) \rightarrow W\) for some \(V \in \text{Ob}C_X\).

Notice that the condition (b) is fulfilled if the category \(C_X\) has an initial object (for instance, \(C_X\) is (pre)additive). In fact, every functor having a right adjoint maps an initial object to initial object. In particular, each of the inverse image functors, \(p_1^*\) and \(p_2^*\), maps an initial object, \(\bullet\), of \(C_X\) to an initial object of \(C_{\mathfrak{Y}}\). Since all initial objects are isomorphic, there is an isomorphism \(\phi: p_1^*(\bullet) \xrightarrow{\sim} p_2^*(\bullet)\). In other words, \((\bullet, \phi)\) is an (initial) object of the category \(C_X\) which \(f^*\) maps to \(\bullet\), an initial object of \(C_{\mathfrak{Y}}\), hence the condition (b).

C8.3. **Example: quasi-compact quasi-separated schemes.** Let \(\mathfrak{X}\) be a quasi-compact, quasi-separated scheme or algebraic space. Then \(\mathfrak{X}\) has a finite affine cover.
\{U_i \to X \mid i \in J\} and for any \(i,j \in J\), the intersection \(U_i \cap U_j\) has a finite affine cover \(\{U_{ij}^k \to U_i \cap U_j \mid k \in J_{ij}\}\). Set \(U = \coprod_{i \in J} U_i\) and \(R = \coprod_{i,j \in J} \coprod_{k \in J_{ij}} U_{ij}^k\). Then we have an exact diagram of schemes (resp. algebraic spaces)

\[
R \longrightarrow U \longrightarrow X
\]

Since \(R\) and \(U\) are affine schemes, the projections \(R \xrightarrow{p_1} U \xrightarrow{p_2} X\) are affine morphisms. By construction, inverse image functors of \(p_1\) and \(p_2\) are conservative and exact, as well as the functor \(\pi^*\). As any morphism to a quasi-separated quasi-compact scheme, \(\pi\) has a direct image functor (this, by the way, follows from the argument of C8.2.1) which makes \(\pi\) (or rather the corresponding morphism of categories of quasi-coherent sheaves) satisfy the conditions of Beck’s theorem; hence \(\text{Qcoh}_X\) is equivalent to the category of \(G_\pi\)-comodules, where \(G_\pi = (\pi^*\pi^*, \delta_\pi)\) is the comonad associated with the continuous morphism \(\pi\). This, however, does not imply that the morphism \(U \xrightarrow{\pi} X\) is affine.

The morphism \(\pi\) being not affine is equivalent to any of the following conditions:
(a) the direct image functor \(\pi_*\) is not right exact;
(b) the comonad \(G_\pi\) is not continuous;
(c) the functor \(G_\pi = \pi^*\pi_*\) is not exact;
(d) the functor \(F_\pi = \pi_*\pi^*\) is not exact.

This follows from the equivalence of the following conditions on a scheme morphism \(\pi\):
(i) \(\pi\) is affine (in the conventional sense);
(ii) its direct image functor is exact (Serre’s criterion);
(iii) \(\pi\) induces an affine morphism of categories of quasi-coherent sheaves.

Notice, however, that the flatness of the morphism \(\pi\) (i.e. the exactness of \(\pi^*\)) is equivalent to that the functor \(G_\pi\) is left exact.

Thus, for any quasi-compact quasi-separated scheme or algebraic space \(X\), the object \(|\text{Qcoh}_X|\) of \(|\text{Cat}|^o\) corresponding to the category of quasi-coherent sheaves on \(X\) is isomorphic to the cospectrum of a left exact comonad on \(\text{Sp}(R)\) for some (commutative) ring \(R\) which is not, in general, given by a coalgebra in the category of \(R\)-bimodules.

\textbf{C9. Monads, comonads, and relations.} For any object \(X\) of the category \(|\text{Cat}|^o\), we denote by \(\mathcal{M}\text{on}_X\) the category of monads on \(X\) and by \(\mathcal{C}\mathcal{M}\text{on}_X\) the category of comonads on \(X\). We have two functors

\[
\text{Sp}_X : \mathcal{M}\text{on}_X^{op} \longrightarrow |\text{Cat}|^o/X, \quad (\mathcal{F}/X) \longmapsto (\text{Sp}(\mathcal{F}/X) \longrightarrow X),
\]

and

\[
\text{Sp}_X^o : \mathcal{C}\mathcal{M}\text{on}_X^{op} \longrightarrow X|\text{Cat}|^o, \quad (X\setminus \mathcal{G}) \longmapsto (X \longrightarrow \text{Sp}^o(X\setminus \mathcal{G})).
\]

\textbf{C9.1. Base change.} Let \(X \xrightarrow{f} Y\) be a continuous morphism with an inverse image functor \(f^*\), a direct image functor \(f_*\) and adjunction arrows \(\text{Id}_{C_X} \xrightarrow{\eta_f} f_*f^*\) and \(f^*f_* \xrightarrow{\epsilon_f} \text{Id}_{C_Y}\). Let \(\mathcal{G} = (G, \delta)\) be a comonad on \(Y\) with counit \(\varepsilon\). The comultiplication \(\delta\)
and the adjunction arrow \( \eta_f \) induce a comultiplication \( \delta^f \) on \( G^f = f^*Gf_* \) defined as the composition
\[
G^f = f^*Gf_* \xrightarrow{f^*\delta^f} f^*G^2f_* \xrightarrow{f^*G\eta_fG^f} (f^*Gf_*)^2 = (G^f)^2.
\]
The counit \( \varepsilon \) and the adjunction arrow \( \epsilon_f \) determine the counit, \( \varepsilon^f \), of \( (G^f, \delta^f) \) given by the composition
\[
G^f = f^*Gf_* \xrightarrow{f^*\varepsilon_f} f^*f_* \xrightarrow{\epsilon_f} \text{Id}_{C_X}.
\]

**C9.1.1. Lemma.** Let \( X \xrightarrow{f} Y \) be a continuous morphism with inverse and direct image functors resp. \( f^* \) and \( f_* \) and adjunction arrows \( \text{Id}_{C_X} \xrightarrow{\eta_f} f_*f^* \) and \( f^*f_* \xrightarrow{\epsilon_f} \text{Id}_{C_Y} \). Let \( G = (G, \delta) \) be a comonad on \( Y \) with counit \( \varepsilon \).

(a) For any comonad \( G = (G, \delta) \) on \( Y \) having the counit \( \varepsilon \), the pair \( G^f = (G^f, \delta^f) \) is a comonad on \( X \) with the counit \( \varepsilon^f \).

(b) The correspondence \( G \mapsto G^f \) extends naturally to a functor \( \mathcal{C}\text{Mon}_Y \to \mathcal{C}\text{Mon}_X \).

(c) For every comonad \( G = (G, \delta) \) on \( Y \), the inverse image functor \( f^* \) induces a functor
\[
f_G^* : (Y\setminus G) - \text{Comod} \longrightarrow (X\setminus G^f) - \text{Comod}
\]
which can be regarded as an inverse image functor of a morphism
\[
f_G : \text{Sp}^0(X\setminus G^f) \longrightarrow \text{Sp}^0(Y\setminus G).
\]

(d) If the category \( C_X \) has kernels of coreflexive pairs of arrows, then the functor (3) has a right adjoint, i.e. the morphism (4) is continuous.

(e) If the comonad \( G = (G, \delta) \) and the morphism \( f \) are weakly flat (i.e. the functors \( G \) and \( f^* \) preserve kernels of coreflexive pairs of arrows), then \( G^f \) is weakly flat too.

**Proof.** (a) The comonad \( G \) can be regarded as the one obtained from an adjoint pair of functors, i.e. \( G = g^*g_* \), \( \delta = g^*\eta_gg_* \), where \( \eta_g \) is an adjunction arrow, and the counit \( \varepsilon \) is a complementary adjunction morphism \( \epsilon_g \) (see 5.3). Then the \( G^f \) becomes the comonad corresponding to the pair of adjoint functors \( f^*g^*, g_*f_* \) and the adjunction morphisms \( g_*\eta_fg^* \circ \eta_g \) and \( \epsilon_f \circ f^*\epsilon_gf_* \).

(b) The functoriality of the map \( G \mapsto G^f \) is evident.

(c) The functor (3) assigns to every \((Y\setminus G)\)-comodule \((L, \xi)\) the \((X\setminus G^f)\)-comodule \((f^*(L), \xi^f)\), where the coaction \( \xi^f : f^*(L) \longrightarrow G^f(f^*(L) = f^*Gf_*f^*(L) \) is the composition \( Gf(L) \circ f^*(\xi) \).

(d) The comonad \( G \) corresponds to an adjoint pair of functors, \( g^* \dashv g_* \), regarded as resp. inverse and direct image functors of a continuous morphism \( Y \xrightarrow{g} Z = \text{Sp}^0(Y\setminus G) \). By (a), \( G^f \) is the comonad corresponding to the adjoint pair \( f^*g^*, g_*f_* \) of resp. inverse and direct image functors of the continuous morphism \( gX \xrightarrow{f} Z \). By 5.4, the inverse image functor \( f^*g^* \) of \( gf \) decomposes canonically into
\[
C_Z = (Y\setminus G) - \text{Comod} \xrightarrow{f_G^*} (X\setminus Gf) - \text{Comod} \longrightarrow C_Y.
\]
Since $C_X$ has kernels of coreflexive pairs of arrows, it follows from 5.4.1(a) (Beck’s theorem) that the functor $f^*_G$ in this decomposition has a right adjoint.

(e) If the functors $f^*$ and $G$ preserve limits of certain type, the functor $G^f = f^*Gf_*$ does the same, since $f_*$ preserves limits of any small diagram. □

**C9.2. Morphisms** $(f \lambda)$. Let $X, Y$ be objects of $|\text{Cat}|^o$; and let $\mathcal{G}$ and $\mathcal{H}$ be comonads resp. on $X$ and $Y$. A morphism $(Y \mathcal{H}) \to (X \mathcal{G})$ is a pair $(f \lambda)$, where $f$ is a continuous morphism $X \to Y$, $\lambda$ is a comonad morphism $(X \mathcal{H}^f) \to (X \mathcal{G})$. It follows from this definition that the morphism $(f \lambda)$ is decomposed into a morphism of underlying objects and a morphism of comonads:

$$
\begin{array}{ccc}
(Y \mathcal{H}) & \xrightarrow{(f \lambda)} & (X \mathcal{G}) \\
(f \mathcal{H}^f) \searrow & & \nearrow (X \mathcal{G})\\
(X \mathcal{H}^f)
\end{array}
$$

(5)

Here $(X \lambda) = (id_X \lambda)$ and $(f \mathcal{H}^f) = (f \lambda Hf)$. Any morphism $(f \lambda) : (X \mathcal{G}) \to (Y \mathcal{H})$ induces a functor

$$
|f \lambda|^* : (Y \mathcal{H}) - \text{Comod} \longrightarrow (X \mathcal{G}) - \text{Comod}
$$

(6)

which is the composition of the functor

$$
f^*_H : (Y \mathcal{H}) - \text{Comod} \longrightarrow (X \mathcal{H}^f) - \text{Comod}
$$

(7)

(cf. C9.1) and the inverse image

$$
\lambda^* : (X \mathcal{H}^f) - \text{Comod} \longrightarrow (X \mathcal{G}) - \text{Comod}
$$

(8)

of the morphism $\lambda$ (see (5) above and 5.3.2). The functor $|f \lambda|^*$ is regarded as an inverse image functor of a morphism

$$
\text{Sp}^o(f \lambda) : \text{Sp}^o(X \mathcal{G}) \longrightarrow \text{Sp}^o(Y \mathcal{H}).
$$

(9)

We shall use also a short-hand notation $|f \lambda|$ instead of $\text{Sp}^o(f \lambda)$.

**C9.2.1. Proposition.** Let $\mathcal{G} = (G, \delta_G)$ and $\mathcal{H} = (H, \delta_H)$ be comonads on resp. $Y$ and $X$, and let $(f \lambda) : (X \mathcal{G}) \to (Y \mathcal{H})$ be a comonad morphism. If the category $C_X$ has kernels of coreflexive pairs of arrows and the functors $f^*$ and $H$ preserve these kernels, then the morphism $\text{Sp}^o(f \lambda)$ is continuous, i.e. the functor $|f \lambda|^*$ has a right adjoint.

**Proof.** By definition, the functor $|f \lambda|^*$ is the composition of the functors (7) and (8). By C9.1(d), the functor (7) has a right adjoint if $C_X$ has kernels of coreflexive pairs of arrows. By 5.3.2.2, the functor (8) has a right adjoint if the category $C_X$ has kernels of coreflexive pairs of arrows and the functor $H^f$ preserves these kernels. The latter condition holds if $H$ and $f$ have this property. □
C9.3. Proposition. Let

\[(Y \backslash \mathcal{H}) \xrightarrow{(p \backslash \lambda)} (X \backslash \mathcal{G}) \xleftarrow{(q \backslash \gamma)} (X \backslash \mathcal{H}^q)\]

be comonad morphisms such that

(i) The pull-back, \((X \backslash \mathcal{H}^p \times_{\lambda, \gamma} \mathcal{H}^q)\), of comonad morphisms

\[
\begin{array}{ccc}
(X \backslash \mathcal{H}^p) & \xrightarrow{(X \backslash \lambda)} & (X \backslash \mathcal{G}) \\
(p \backslash \mathcal{H}^p) & \downarrow & (X \backslash \mathcal{H}^q) \\
(Y \backslash \mathcal{H}) & & (Y \backslash \mathcal{H}^q)
\end{array}
\]

exists.

(ii) The category \(C_Y\) has kernels of coreflexive pairs of arrows, and the morphisms \(X \xrightarrow{p} Y\) are weakly flat and conservative.

Then the pushforward of

\[
\begin{array}{c}
\text{Sp}^\circ(Y \backslash \mathcal{H}) \\
\text{Sp}^\circ(p \backslash \lambda)
\end{array} \quad \begin{array}{c}
\text{Sp}^\circ(X \backslash \mathcal{G}) \\
\text{Sp}^\circ(q \backslash \gamma)
\end{array} \quad \begin{array}{c}
\text{Sp}^\circ(Y \backslash \mathcal{H}) \\
\text{Sp}^\circ(X \backslash \mathcal{H}^q)
\end{array}
\]

is naturally isomorphic to \(\text{Sp}^\circ(X \backslash \mathcal{H}^p \times_{\lambda, \gamma} \mathcal{H}^q)\).

Proof. The pair of morphisms \((p \backslash \lambda), (q \backslash \gamma)\) is described by the diagram

\[
\begin{array}{ccc}
(X \backslash \mathcal{H}^p) & \xrightarrow{(X \backslash \lambda)} & (X \backslash \mathcal{G}) \\
(p \backslash \mathcal{H}^p) & \downarrow & (X \backslash \mathcal{H}^q) \\
(Y \backslash \mathcal{H}) & & (Y \backslash \mathcal{H}^q)
\end{array}
\]

If the condition (a) holds, i.e. a pull-back of the pair of comonad morphisms \(\lambda, \gamma\) exists, we complete (4) to a commutative diagram

\[
\begin{array}{ccc}
(X \backslash \mathcal{G}) & \xrightarrow{(X \backslash \gamma)} & (X \backslash \mathcal{H}^q) \\
(X \backslash \lambda) & \downarrow & (X \backslash \pi_q) \\
(Y \backslash \mathcal{H}) & \xrightarrow{(p \backslash \mathcal{H}^p)} & (X \backslash \mathcal{H}^p) \\
& & \xrightarrow{(X \backslash \pi_p)} (X \backslash \mathcal{H}^p \times_{\lambda, \gamma} \mathcal{H}^q)
\end{array}
\]

Applying the functor \(\text{Sp}^\circ\) to (5), we obtain the diagram

\[
\begin{array}{ccc}
\text{Sp}^\circ(X \backslash \mathcal{G}) & \xrightarrow{\text{Sp}^\circ(X \backslash \gamma)} & \text{Sp}^\circ(X \backslash \mathcal{H}^q) \\
\text{Sp}^\circ(Y \backslash \mathcal{H}) & \xrightarrow{\text{Sp}^\circ(p \backslash \mathcal{H}^p)} & \text{Sp}^\circ(X \backslash \mathcal{H}^p) \\
& \downarrow & \text{Sp}^\circ(X \backslash \pi_q) \\
& \downarrow & \text{Sp}^\circ(X \backslash \pi_p) \\
\text{Sp}^\circ(Y \backslash \mathcal{H}) & \xrightarrow{\text{Sp}^\circ(q \backslash \mathcal{H}^q)} & \text{Sp}^\circ(X \backslash \mathcal{H}^p \times_{\lambda, \gamma} \mathcal{H}^q)
\end{array}
\]

By condition (b) and Beck’s theorem, both \(\text{Sp}^\circ(p \backslash \mathcal{H}^p)\) and \(\text{Sp}^\circ(q \backslash \mathcal{H}^q)\) are isomorphisms, hence the assertion.
C9.3.1. Corollary. Under the conditions of C9.3, the cokernel of
\[ \text{Sp}^\circ(Y \bslash \mathcal{H}) \xrightarrow{\text{Sp}^\circ(p \bslash \lambda)} \text{Sp}^\circ(X \bslash \mathcal{G}) \]
is naturally isomorphic to the cokernel of
\[ \text{Sp}^\circ(X \bslash \mathcal{H}^p) \xrightarrow{\text{Sp}^\circ(p \bslash \lambda)} \text{Sp}^\circ(X \bslash \mathcal{H}^p \times_{\lambda, \gamma} \mathcal{H}^q), \]
where one arrow is \( \text{Sp}^\circ(X \bslash \pi_p) \) and another arrow is the composition of
\[ \text{Sp}^\circ(X \bslash \pi_q) : \text{Sp}^\circ(X \bslash \mathcal{H}^q) \longrightarrow \text{Sp}^\circ(X \bslash \mathcal{H}^p \times_{\lambda, \gamma} \mathcal{H}^q) \]
and the isomorphisms
\[ \text{Sp}^\circ(X \bslash \mathcal{H}^p) \xrightarrow{\text{Sp}^\circ(p \bslash \lambda)^{-1}} \text{Sp}^\circ(Y \bslash \mathcal{H}) \quad \text{and} \quad \text{Sp}^\circ(Y \bslash \mathcal{H}) \xrightarrow{\text{Sp}^\circ(q \bslash \lambda)} \text{Sp}^\circ(X \bslash \mathcal{H}^q) \]
(see the diagram (6)).

Proof. The assertion follows from the diagram (6) and the general nonsense fact about the connection of the push-forward of a pair of arrows \( x \xleftarrow{\alpha} y \xrightarrow{\beta} x \) and the cokernel of this pair. ■

C9.4. The case of affine covers and relations. Let
\[ \text{Sp}(R) \xrightarrow{p} \mathfrak{A} \xrightarrow{q} X \]
be an exact diagram in \(|\text{Cat}|^\circ\) such that all morphisms are faithfully flat. To the pair of morphisms \( p, q \), there corresponds the pair of morphisms
\[ (R \bslash \mathcal{H}^p) \xrightarrow{(R \bslash \mu_p)} (R \bslash \mathcal{H}^q) \xleftarrow{(R \bslash \mu_q)} (R \bslash \mathcal{H}^q) \]
(cf. C9.3(a)), where \( \mathcal{H}^p \) is \((R \otimes_{(A, p)} R, \delta_p)\) and \( \mathcal{H}^q = (R \otimes_{(A, q)} R, \delta_q)\), and morphisms \( \mu_p \) and \( \mu_q \) are induced by the multiplication \( R \otimes R \longrightarrow R \). The pull-back of (2) is the coalgebra \((R \bslash \mathcal{H}^p)\). Here \( H_{p,q} = (R \otimes R)/(I_p \cap I_q) \), where \( I_p \) (resp. \( I_q \)) is the kernel of the epimorphism \( R \otimes R \longrightarrow R \otimes_{(A, p)} R \) (resp. \( R \otimes R \longrightarrow R \otimes_{(A, q)} R \)); and the comultiplication \( \delta_{p,q} \) is uniquely determined by the condition that the natural epimorphism \( R \otimes R \longrightarrow H_{p,q} \) is a coalgebra morphism \((R \bslash (R \otimes R, \delta_R)) \longrightarrow (R \bslash (H_{p,q}, \delta_{p,q}))\).

Set \( \mathcal{H}_{p,q} = (H_{p,q}, \delta_{p,q}) \). It follows from the construction that the square
\[ (R \bslash \mathcal{H}_{p,q}) \longrightarrow (R \bslash \mathcal{H}^p) \]
\[ \downarrow \hspace{2cm} \downarrow \]
\[ (R \bslash \mathcal{H}^q) \longrightarrow (R \bslash \mathcal{H}^q) \]
\[ \text{is commutative.} \]
commutes. This implies that there is a natural morphism $\text{Sp}\mathfrak{A} \longrightarrow \text{Sp}^\circ(R \setminus \mathcal{H}_{p,q})$ which equalizes the pair of arrows $\text{Sp}(R) \overset{p}{\longrightarrow} \text{Sp}\mathfrak{A}$. This morphism is the composition of the morphism $\text{Sp}^\circ(R \setminus \mathcal{H}^p) \longrightarrow \text{Sp}^\circ(R \setminus \mathcal{H}_{p,q})$ induced by the coalgebra morphism $(R \setminus \mathcal{H}_{p,q}) \longrightarrow (R \setminus \mathcal{H}^p)$ and the isomorphism $\text{Sp}\mathfrak{A} \longrightarrow \text{Sp}^\circ(R \setminus \mathcal{H}^p)$.

It is not true in general that the functor $\text{Sp}^\circ$ transforms (3) into a cocartesian square, or, equivalently, that $\text{Sp}^\circ(R \setminus \mathcal{H}_{p,q})$ is a cokernel of the pair of arrows $\text{Sp}(R) \overset{p}{\longrightarrow} \text{Sp}\mathfrak{A}$.

C9.4.1. Proposition. The square (3) is cocartesian, or, equivalently, the diagram

$$\text{Sp}(R) \overset{p}{\longrightarrow} \text{Sp}\mathfrak{A} \longrightarrow \text{Sp}^\circ(R \setminus \mathcal{H}_{p,q}) \quad (4)$$

is exact, iff $H_{p,q}$ is flat as a right $R$-module.

Proof. The fact follows from C9.3 (and holds in a more general situation). Details are left to the reader. ■

C9.4.2. Remark. The realization of the cokernel of the pair $\text{Sp}(R) \overset{p}{\longrightarrow} \text{Sp}\mathfrak{A}$ as $\text{Sp}^\circ(R \setminus \mathcal{H}_{p,q})$ (under the conditions of C9.4.1) might be not the best choice. For instance, suppose that the object $X$ in the exact sequence (1) is also affine, i.e. $X = \text{Sp}(S)$ for some ring $S$. In this case, we can assume (after replacing some of the rings in (1) by Morita equivalent rings) that the diagram (1) is obtained by applying the functor $\text{Sp}$ to the exact diagram of rings $S \overset{\phi}{\longrightarrow} \mathfrak{A} \overset{p}{\longrightarrow} R$. It follows (from Beck’s theorem) that $\text{Sp}(S) \simeq \text{Sp}^\circ(\mathfrak{A} \setminus \mathfrak{A} \otimes_S \mathfrak{A})$.

C9.5. Quasi-coherent modules on a Grassmannian. Let $R \overset{\phi}{\longrightarrow} S$ be a $k$-algebra morphism. Let $M$ be a left $R$-module, and let $L$, $N$ be projective left $R$-modules of finite type. We have canonical isomorphisms

$$\text{Hom}_S(S \otimes_R M, S \otimes_R L) \simeq \text{Hom}_R(M, S \otimes_R L) \simeq \text{Hom}_{R^e}(M \otimes_k L^\vee_R, S) \quad (1)$$

(cf. the argument of [KR3, C9.1.3]). Here $R^e = R \otimes_k R^{\text{op}}$. In particular, the composition,

$$\text{Hom}_S(S \otimes_R M, S \otimes_R L) \times \text{Hom}_S(S \otimes_R L, S \otimes_R N) \overset{\epsilon_{M,L,N}^S}{\longrightarrow} \text{Hom}_S(S \otimes_R M, S \otimes_R N),$$

determines a unique map

$$\text{Hom}_{R^e}(M \otimes_k L^\vee_R, S) \times \text{Hom}_{R^e}(L \otimes_k N^\vee_R, S) \overset{\lambda_{M,L,N}}{\longrightarrow} \text{Hom}_{R^e}(M \otimes_k N^\vee_R, S) \quad (2)$$

The map $\lambda_{M,L,N}$ assigns to any pair of $R^e$-module morphisms, $M \otimes_k L^\vee_R \overset{u}{\longrightarrow} S$ and $L \otimes_k N^\vee_R \overset{v}{\longrightarrow} S$, the composition

$$M \otimes_k N^\vee_R \longrightarrow M \otimes_k L^\vee_R \otimes_R L \otimes_k N^\vee_R \overset{u \otimes_R v}{\longrightarrow} S \otimes_R S \longrightarrow S.$$
Here the first arrow is induced by the canonical map $k \rightarrow L_R^\vee \otimes_R L$ (the composition of the morphism $k \rightarrow Hom_R(L, L)$ sending the unit of $k$ to the identity morphism, $id_L$, and the isomorphism $Hom_R(L, L) \rightarrow L_R^\vee \otimes_R L$) and the last arrow is induced by the multiplication on $S$.

The relation $u \circ v = id_{S \otimes_R L}$ (defining the functor $G_{m,l}$) is expressed by the commutative diagram

\[
\begin{array}{ccc}
L \otimes_k M_R^\vee \otimes_R M \otimes_k L_R^\vee & \overset{v \otimes_R u}{\rightarrow} & S \otimes_R S \\
\lambda_{L,M,L} \uparrow & & \downarrow \\
L \otimes L_R^\vee & \overset{ev_L}{\rightarrow} & R \overset{\phi}{\rightarrow} S
\end{array}
\]\n
(2)

Here $ev_L$ denotes the evaluation morphism, $a \otimes \lambda \mapsto \langle \lambda, a \rangle$.

Taking as $S$ the $R$-ring $T(L \otimes_k M_R^\vee \otimes_k M \otimes_k L_R^\vee)$ representing the functor

\[R \setminus Alg_k \rightarrow Sets, \quad U \mapsto Hom_U(U \otimes_R L, U \otimes_R M) \times Hom_U(U \otimes_R M, U \otimes_R L),\]

and as $u$, $v$ canonical $R$-bimodule morphisms resp. $L \otimes_k M_R^\vee \rightarrow U$ and $M \otimes_k L_R^\vee \rightarrow U$, one might describe an $R$-ring, $G = G_{m,l}$, representing the functor $G_{m,l} : R \setminus Alg_k \rightarrow Sets$ (cf. C7.1) as the colimit of the (non-commutative) diagram (2).

The relations $u_j \circ (v_i \circ u_i) = u_i$, $i, j = 1, 2$, defining (together with relations $u_i \circ v_i = id_{S \otimes_R L}$, $i = 1, 2$) the functor $\mathcal{R}_{m,l}$ are expressed by the commutative diagrams

\[
\begin{array}{ccc}
M \otimes_k M_R^\vee \otimes_R M \otimes_k L_R^\vee \otimes_R L \otimes_k M_R^\vee \otimes_R M \otimes_k L_R^\vee & \overset{u_i \otimes_R v_i \otimes_R u_j}{\rightarrow} & S \otimes_R S \otimes_R S \\
\lambda_{M,L,M} \uparrow & & \downarrow \\
M \otimes M_R^\vee \otimes_R M \otimes_k L_R^\vee & \overset{(v_i \circ u_i) \otimes_R u_j}{\rightarrow} & S \otimes_R S \\
\rightarrow_R M \otimes_k L_R^\vee & \overset{\phi \otimes_R u_j}{\rightarrow} & S \otimes_R S
\end{array}
\]\n
(3)

Fix an $R$-ring $G = G_{m,l}$ corepresenting the functor $G_{m,l} : R \setminus Alg_k \rightarrow Sets$. And let \[G \otimes_R L \overset{v}{\rightarrow} G \otimes_R M \overset{u}{\rightarrow} G \otimes_R L\] be a canonical splitting — the image of $id_G$ under the isomorphism $Alg(G, G) \rightarrow G_{m,l}(G)$. Set $p = v \circ u : G \otimes_R M \rightarrow G \otimes_R M$. Denote by $H$ a colimit of the diagram

\[
\begin{array}{ccc}
M \otimes_k M_R^\vee \otimes_R M \otimes_k L_R^\vee \otimes_R L \otimes_k M_R^\vee \otimes_R M \otimes_k L_R^\vee & \overset{p \otimes_R u}{\rightarrow} & G \otimes_R G \\
\lambda_{M,M,R,L} \uparrow & & \downarrow \\
M \otimes L_R^\vee \otimes_R M \otimes_k L_R^\vee & \overset{\phi \otimes_R u}{\rightarrow} & G \otimes_R G
\end{array}
\]\n
(4)

Note that $H$ is a quotient of the bimodule $G \otimes_R G$. The comultiplication, $\delta$, on $H$ is uniquely defined by the condition: the canonical $G$-bimodules epimorphism $G \otimes_R G \rightarrow H$ is a coalgebra morphism.
The category of quasi-coherent sheaves on the locally affine space $Gr_{M,L}^+$ is equivalent to the category $(H, \delta) - \text{comod}$ of $(H, \delta)$-comodules.

C9.5.1. Example: the zero dimensional projective space. Let $R = k$. The zero dimensional projective space is by definition $\mathbb{P}^1_k = Gr_{k^1, k^1}^+$. The algebra $G$ representing the functor $\mathbb{P}^1_k$ is the quotient of the free $k$-algebra $k\langle x, y \rangle$ in two variables by the two-sided ideal generated by $xy - 1$: $G = k\langle x, y \rangle/(xy - 1)$. Denote by $e$ the element $1 - yx$ of $G$. It follows that $e$ is an idempotent, $e^2 = e$. The coalgebra corresponding to the projection $\pi : \text{Spec}(A) \rightarrow \mathbb{P}^1_k$ is $(H, \delta)$, where the bimodule $H$ enters into the exact sequence

$$0 \rightarrow G \otimes eG \xrightarrow{i} G \otimes G \rightarrow H \rightarrow 0 \quad (1)$$

Here $\otimes = \otimes_k$, and $i$ is defined by $a \otimes eb \mapsto ax \otimes eb$. The comultiplication $\delta$ is uniquely determined by the commutativity of the diagram

$$\begin{array}{ccc}
H & \xrightarrow{\delta} & H \otimes G H \\
\uparrow & & \uparrow \\
G \otimes G & \xrightarrow{\delta^\sim} & G \otimes G \otimes G
\end{array}$$

where $\delta^\sim$ sends $a \otimes b$ to $a \otimes 1 \otimes b$. The counit $\epsilon : H \rightarrow G$ is induced by the multiplication $G \otimes G \rightarrow G$ in the algebra $G$.

C10. Miscellaneous.

C10.1. Connections. Let $X \xrightarrow{f} Y$ be a continuous morphism. Fix inverse and direct image functors of $f$ and adjunction arrows, $Id_{C_Y} \xrightarrow{\eta_f} f_*f^*$ and $f^*f_* \xrightarrow{\epsilon_f} Id_{C_X}$. Let $G_f$ be the comonad associated with this data: $G_f = (G_f, \delta_f)$, where $G_f = f^*f_*$ and $\delta_f = f^*\eta_ff_*$.

A $(X \setminus G_f)$-connection is a pair $(M, \rho)$, where $M$ is an object of $C_X$ and $\rho$ a morphism $M \rightarrow G_f(M)$ such that $\epsilon_f(M) \circ \rho = \text{id}_M$. A morphism from an $f$-connection $(M, \rho)$ to an $f$-connection $(M', \rho')$ is a morphism $g : M \rightarrow M'$ such that the diagram

$$\begin{array}{ccc}
M & \xrightarrow{g} & M' \\
\rho \downarrow & & \downarrow \rho' \\
G_f(M) & \xrightarrow{G_f(g)} & G_f(M')
\end{array}$$

commutes. Composition is defined in a natural way. We denote the category of $G_f$-connections by $\mathcal{C} \mathfrak{onn}(X \setminus G_f)$.

A $G_f$-connection $(M, \rho)$ is called integrable if it is an $(X \setminus G_f)$-comodule. Thus, the category $(X \setminus G) - \text{Comod}$ is a full subcategory of the category $\mathcal{C} \mathfrak{onn}(X \setminus G_f)$ formed by connections and morphisms between them.

C10.1.1. Connections and monadic morphisms. Suppose that the morphism $X \xrightarrow{f} Y$ is monadic, i.e. the arrow $\bar{f}$ in the diagram 5.4(3) is an isomorphism. Then we can
assume (for our immediate purposes) that \( X = \text{Sp}(\mathcal{F}_f/Y) \) and take the standard inverse and direct image functors of the canonical morphism \( f : \text{Sp}(\mathcal{F}_f/Y) \rightarrow Y \). The functor \( G_f : (\mathcal{F}_f/Y) - \text{mod} \rightarrow (\mathcal{F}_f/Y) - \text{mod} \) assigns to each \((\mathcal{F}_f/Y)\)-module \( M = (M, \xi) \) the \((\mathcal{F}_f/Y)\)-module \((F_f(M), \mu_f(M))\). The comultiplication \( \delta_f : G_f \rightarrow G^2_f \) is \( F_f \eta_f \), where \( \eta_f \) is the unit of the monad \( \mathcal{F}_f \); the counit \( \epsilon_f : G_f \rightarrow \text{Id}_{(\mathcal{F}_f/Y) - \text{mod}} \) is defined by \( \epsilon_f(M, \xi) = \xi : F_f(M) \rightarrow M \).

A \((X \setminus \mathcal{G}_f)\)-connection is a pair \((\mathcal{M}, \rho)\), where \( \mathcal{M} = (M, \xi) \) is an \((\mathcal{F}_f/Y)\)-module and \( \rho \) an \((\mathcal{F}_f/Y)\)-module morphism \((M, \xi) \rightarrow (F_f(M), \mu_f(M))\) such that \( \xi \circ \rho = \text{id}_M \). In other words, \( \rho \) is a morphism \( M \rightarrow F_f(M) \) satisfying the equations

\[
\mu_f(M) \circ F_f(\rho) = \rho \circ \xi, \quad \xi \circ \rho = \text{id}_M.
\]

The connection \((\mathcal{M}, \rho)\) is integrable iff \( F_f \eta_f(M) \circ \rho = F_f(\rho) \circ \rho \). It follows from (1) that \((M, \xi)\) is the kernel of the pair of arrows

\[
F_f \eta_f(M), F_f(\rho) : F_f(M) \longrightarrow F^2_f(M)
\]

and \( \rho : (M, \xi) \rightarrow \mathcal{F}_f(M) = (F_f(M), \mu_f(M)) \) is a universal arrow. The pair

\[
\eta_f(M), \rho : M \longrightarrow F_f(M)
\]

is coreflexive, since \( \xi \circ F_f \eta_f(M) = \text{id}_M = \xi \circ \rho \). Suppose \( C_Y \) has kernels of coreflexive pairs, and let \( L \) be a kernel of the pair (3). If the functor \( F_f \) preserves the kernels of coreflexive pairs, then \( \mathcal{F}_f(L) = (F_f(L), \mu_f(L)) \) is a kernel of the pair (2) which implies that \((M, \xi)\) is isomorphic to \( \mathcal{F}_f(L) \).

**C10.2. Weakly quasi-affine morphisms.** We call a continuous morphism \( X \xrightarrow{f} Y \) weakly quasi-affine if the canonical diagram

\[
(G_f)^2 \xrightarrow{f \circ G_f} G_f \xrightarrow{\epsilon_f} \text{Id}_{C_X}
\]

is exact. Here \( G_f = f^*f_* \) and \( \epsilon_f \) is an adjunction arrow.

**C10.2.1. Proposition.** Let \( Y \in \text{Ob}\{\text{Cat}\}^o \) be such that the category \( C_Y \) has cokernels of reflexive pairs of arrows. The following conditions on a continuous morphism \( X \xrightarrow{f} Y \) are equivalent:

(a) The canonical morphism \( \bar{X} \xrightarrow{f} \text{Sp}(\mathcal{F}_f/Y) \) (cf. 5.4.2) is a localization.

(b) The morphism \( f \) is weakly quasi-affine.

**Proof.** Since the category \( C_Y \) has cokernels of reflexive pairs of arrows, the functor \( \bar{f}_* : C_Y \rightarrow (\mathcal{F}_f/Y) \rightarrow \text{mod}, L \mapsto (f_*(L), f_*\epsilon_f(L)) \), has a left adjoint, \( \bar{f}^* \), which assigns to each \((\mathcal{F}_f/Y)\)-module \((M, \xi)\) a cokernel of the pair of arrows

\[
\begin{array}{c}
f^*f_*(M) \\
\xrightarrow{f^*(\xi)} \end{array} f^*(M).
\]

100
The exactness of the diagram (1) means precisely that the adjunction morphism \( \epsilon_f : f^* \tilde{f} \rightarrow Id_{C_X} \) is an isomorphism. The latter is equivalent to that \( f_* \) is a fully faithful functor, i.e. \( f^* \) is a localization.

C10.2.2. Remark. If a morphism \( X \xrightarrow{f} Y \) is weakly quasi-affine, then an adjunction arrow \( \epsilon_f : f^* f_* \rightarrow Id_{C_X} \) is a strict epimorphism. Notice that \( \epsilon_f \) is an epimorphism if the functor \( f_* \) is faithful. The exactness of the diagram (1) implies that \( f_* \) is conservative. On the other hand, if the functor \( f_* \) reflects cokernels of reflexive pairs of arrows, then the diagram (1) is exact, i.e. \( f \) is weakly quasi-affine. This follows from the observation that the pair of morphisms \( (f^* f_*)^2 \xrightarrow{\epsilon f \tilde{f}^*} f^* f_* \) is reflexive, and the diagram \( f_*(f^* f_*)^2 \xrightarrow{f_* \epsilon f \tilde{f}^*} f_* f^* f_* \rightarrow f_* \),

is exact for any pair of adjoint functors \( f_*, f^* \) and an adjunction morphism \( \epsilon_f \).

By Beck’s theorem, a morphism \( X \xrightarrow{f} Y \) satisfying the equivalent conditions of C10.2.1 is monadic if its direct image functor \( f_* \) preserves cokernels of reflexive pairs of arrows.

C10.2.3. Weakly quasi-affine morphisms to the spectrum of a ring. Let \( R \) be an associative unital ring and \( f \) a continuous morphism \( X \rightarrow \text{Sp}(R) \) with an inverse image functor \( f^* \) and a direct image \( f_* ; \) and let \( \mathcal{O} = f^*(R) \). Consider the decomposition

\[
\begin{array}{ccc}
C_X & \xrightarrow{\hat{f}_*} & \Gamma_X \mathcal{O} - \text{mod} \\
\downarrow f_* & \nearrow \phi_* & \\
R - \text{mod}
\end{array}
\]

(3)

Here \( \Gamma_X \mathcal{O} \) denotes the ring \( C_X(\mathcal{O}, \mathcal{O})^o \), \( \phi_* \) is the pull-back functor by the ring morphism \( \phi : R \rightarrow \Gamma_X \mathcal{O} \) defining a right \( R \)-module structure on \( \mathcal{O} \), and \( \hat{f}_* = C_X(\mathcal{O}, -) \) (see 4.5).

C10.2.3.1. Proposition. Let \( C_X \) be a Grothendieck category and \( R \) an associative unital ring.

(a) A continuous morphism \( X \xrightarrow{f} \text{Sp}(R) \) is weakly quasi-affine iff its direct image functor is faithful.

(b) Every weakly quasi-affine morphism \( X \rightarrow \text{Sp}(R) \) is a composition of an affine morphism and a flat localization.

Proof. By C10.2.2, direct image functor of a weakly quasi-affine morphism is faithful. Conversely, if \( f_* \) is a faithful functor and \( C_X \) is a Grothendieck category, then, by 4.8.2, the canonical morphism \( X \rightarrow \text{Sp}(\Gamma_X \mathcal{O}) \) is a flat localization. Thus the first arrow in the canonical decomposition \( X \rightarrow \text{Sp}(\Gamma_X \mathcal{O}) \rightarrow \text{Sp}(R) \) of the morphism \( f \) is a flat localization and the second one an affine morphism. \( \square \)
C10.2.3.2. Two decompositions of a continuous morphism to the spectrum of a ring. Let \( X \xrightarrow{f} \text{Sp}(R) \) be continuous morphism to the spectrum of a unital associative ring. We have two canonical decompositions of the morphism \( f \) incorporated in the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f_0} & \text{Sp}(\Gamma_X \mathcal{O}) \\
\downarrow \phi_f & & \downarrow \\
\text{Sp}(\mathcal{F}_f/R) & \xrightarrow{f} & \text{Sp}(R)
\end{array}
\]  

(4)

where \( \mathcal{F}_f \) is the monad \((f_* f^*, \mu_f)\) associated with \( f \). Let \( \Gamma_X \mathcal{O}^\sim \) denote the monad on \( \text{Sp}(\mathcal{R}) \) (i.e. on \( \mathcal{R} - \text{mod} \)) determined by the ring morphism \( \phi_f \). Then we have a natural isomorphism

\[
\text{Sp}(\Gamma_X \mathcal{O}^\sim/R) \xrightarrow{\sim} \text{Sp}(\Gamma_X \mathcal{O})
\]

which makes the diagram

\[
\begin{array}{ccc}
\text{Sp}(\Gamma_X \mathcal{O}^\sim/R) & \xrightarrow{\sim} & \text{Sp}(\Gamma_X \mathcal{O}) \\
\downarrow \phi_f & & \downarrow \\
\text{Sp}(\mathcal{R}) & &
\end{array}
\]  

(5)

commute. The monad \( \Gamma_X \mathcal{O}^\sim \) is exactly the continuous monad associated with the monad \( \mathcal{F}_f \) (cf. C2.1). Thus, there exists a canonical monad morphism \( \Gamma_X \mathcal{O}^\sim \xrightarrow{\sim} \text{Sp}(\Gamma_X \mathcal{O}) \) which makes the diagram

\[
\begin{array}{ccc}
\text{Sp}(\Gamma_X \mathcal{O}^\sim/R) & \xrightarrow{\sim} & \text{Sp}(\Gamma_X \mathcal{O}) \\
\downarrow \phi_f & & \downarrow \\
\text{Sp}(\mathcal{R}) & &
\end{array}
\]  

(6)

Adjoining the morphism (6) (or rather the composition of (6) with the isomorphism \( \text{Sp}(\Gamma_X \mathcal{O}^\sim/R) \xrightarrow{\sim} \text{Sp}(\Gamma_X \mathcal{O}) \)), we obtain a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f_0} & \text{Sp}(\Gamma_X \mathcal{O}) \\
\downarrow \phi_f & & \downarrow \\
\text{Sp}(\mathcal{F}_f/R) & \xrightarrow{f} & \text{Sp}(R)
\end{array}
\]  

(7)

If \( f \) is weakly quasi-affine, then, by C10.2.1, the morphism \( \tilde{f} \) is a localization. If, in addition, \( C_X \) is a Grothendieck category, then the morphism \( f_0 \) is a flat localization.

The canonical morphism \( \lambda_f : \text{Sp}(\mathcal{F}_f/R) \xrightarrow{\sim} \text{Sp}(\Gamma_X \mathcal{O}) \) is an isomorphism iff the morphism \( f \) is affine. In this case, \( f_0 : X \xrightarrow{\sim} \text{Sp}(\Gamma_X \mathcal{O}) \) is an isomorphism.

C10.2.4. Relatively ample morphisms. Let \( X \xrightarrow{f} Y \) be a continuous morphism. We call a continuous morphism \( \phi : U \xrightarrow{\sim} X \) \( f \)-ample if the canonical diagram

\[
\begin{array}{ccc}
G^2_{f\phi} & \xrightarrow{\epsilon_{f\phi} G_{f\phi}} & G_{f\phi} \\
\downarrow \epsilon_f G_{f\phi} & & \downarrow \\
G_{f\phi} & \xrightarrow{\epsilon_{f\phi}} & \text{Id}_{C_X} \\
\end{array}
\]  

(1)

(where \( G_{f\phi} = (f\phi)^*(f\phi)_* \) and \( \epsilon_{f\phi} \) is an adjunction arrow) is exact.

It follows that \( \text{id}_X \) is \( f \)-ample iff \( f \) is weakly quasi-affine.
Denote by $U_{f\phi}$ the object of $|\text{Cat}|^o$ such that $C_{U_{f\phi}}$ is the full subcategory of $C_U$ whose objects are all $M \in \text{Ob}C_U$ for which the diagram

$$
G^2_{f\phi}(M) \xrightarrow{\epsilon_{f\phi}G_{f\phi}} G_{f\phi}(M) \xrightarrow{\epsilon_{f\phi}} M
$$

is exact. The condition '$\phi$ is $f$-ample' implies that $\phi$ induces a continuous morphism $\varphi : U_{f\phi} \rightarrow X$ such that $f\varphi$ is weakly quasi-affine.

C10.2.4.1. **Note.** The exactness of the diagram (1) implies that the functor $f_\ast\phi_\ast\phi^*$ is faithful and conservative.

C10.2.4.2. **Example.** Let $X \xrightarrow{f} Y$ be a continuous morphism, and let $\theta : X \rightarrow X$ be a continuous endomorphism. Suppose $C_X$ has countable coproducts. Then $F_\theta = \bigoplus_{n \geq 0}\theta^{\ast n}$ has a natural structure of a monad on $X$. Let $U = \text{Sp}(F_\theta/X)$, and let $\phi = \phi_\theta$ be a canonical morphism $U \rightarrow X$. Thus $\phi_\ast\phi^* = F_\theta$. If $f_\ast$ preserves countable coproducts, $f_\ast\phi_\ast\phi^* \simeq \bigoplus_{n \geq 0}f_\ast\theta^{\ast n}$. In this case, if the morphism $\phi$ is $f$-ample, then the family $\{f_\ast\theta^{\ast n} | n \geq 0\}$ is conservative.

C10.2.4.2.1. **Example.** Let $X = (X, O_X)$ be a scheme and $X \xrightarrow{f} Y$ a quasi-compact scheme morphism. Let $L$ be an invertible sheaf on $X$. Set $\theta = L \otimes O_X \rightarrow -$. The sheaf $L$ is $f$-ample in the usual sense iff the morphism $\phi = \phi_\theta$ constructed in C10.2.4.2 is $f$-ample.

C10.3. **Continuous, affine, and flat morphisms in a fibered category.** Let $\mathcal{P}$ be a subcategory of $|\text{Cat}|^o$, and let $p : \mathcal{F} \rightarrow \mathcal{B}$ be a fibered category. We say that a morphism $f$ of $\mathcal{B}$ belongs to $\mathcal{P}$ if its image by $\mathcal{F}$ belongs to $\mathcal{P}$.

C10.3.1. **Continuous, affine, and flat morphisms in the fibered category of modules on affine schemes.** Consider the fibered category $\mathcal{M}(\mathcal{A}, \mathcal{O})$ over the category $\mathcal{A} = \text{Aff}_k$ of affine $k$-schemes (cf. [KR3, 11.4.1]). All morphisms of $\text{Aff}_k$ are affine. In fact, any morphism $f : \text{Spec}R \rightarrow \text{Spec}S$ is given by an algebra morphism $S \rightarrow R$. By 3.8, the corresponding inverse image functor $f^*$ is $R \otimes S : S-\text{mod} \rightarrow R-\text{mod}$, the direct image functor $f_\ast$ is the 'pull-back' functor $R-\text{mod} \rightarrow S-\text{mod}$, and a right adjoint to $f_\ast$ is given by $f^! : L \rightarrow \text{Hom}_S(R, L)$.

A morphism $f : \text{Spec}R \rightarrow \text{Spec}T$ is flat (resp. faithful) iff the corresponding algebra morphism $T \rightarrow R$ turns $R$ into a flat right $T$-module, that is the functor $R \otimes T$ is exact (resp. faithful).

C10.3.2. **Proposition.** Let $X, Y$ be presheaves of sets on $\text{Aff}_k$ and let $X \xrightarrow{f} Y$ be a morphism of presheaves. Suppose $Y$ is affine (i.e. representable). Then $X$ is affine iff the morphism $|\text{Qcoh}_X| \rightarrow |\text{Qcoh}_Y|$ in $|\text{Cat}|^o$ induced by $f$ is affine.

**Proof.** Let $Y$ be isomorphic to $\text{Spec}R$ for some $k$-algebra $R$. By [KR3, 11.1.5.1], the category $\text{Qcoh}_Y$ is equivalent to the category $R-\text{mod}$.

103
(a) If X is affine, i.e. $X \cong \text{Spec} S$ for a $k$-algebra $S$, then $Qcoh_X$ is equivalent to the category $S - \text{mod}$ and we have a quasi-commutative diagram

$$
\begin{array}{ccc}
S - \text{mod} & \longrightarrow & Qcoh_X \\
\phi^* \uparrow & & \uparrow f^* \\
R - \text{mod} & \longrightarrow & Qcoh_X
\end{array}
$$

where $\phi^*$ is the inverse image functor corresponding to a $k$-algebra morphism $R \to S$; i.e. $\phi^* = S \otimes_R -$. Since $\phi^*$ is an inverse image of an affine morphism (by the argument above or 3.8), $f$ is affine.

(b) The converse assertion follows from 6.6.1.

Let $\text{Ass}_k$ be the category whose objects are associative $k$-algebras. Morphisms from a $k$-algebra $R$ to a $k$-algebra $S$ are equivalence classes of algebra morphisms $R \to S$ by the following equivalence relation: two algebra morphisms, $f, g : R \to S$ are equivalent if they are conjugated, i.e. $g(-)t = tf(-)$ for an invertible element $t$ of $S$.

Note that the restriction of the natural functor $\text{Ass}_k \to \text{Ass}_k$ to the subcategory of commutative algebras is a strict fully faithful functor ('strict' means that it is injective on objects).

**C10.3.3. Proposition.** The canonical functor $\text{Aff}_k \to |\text{Cat}|^o$ induces a faithful functor from the category $\text{Ass}^o_k$ to the subcategory $|\text{Cat}|^o_{\text{aff}}$ of $|\text{Cat}|^o$ formed by affine morphisms.

**Proof.** The assertion is a corollary of [KR3, 5.10.1] and C10.3.2.

Denote by $\mathcal{B}\text{Ass}$ the category whose objects are associative unital rings and morphisms from $R$ to $S$ isomorphism classes of $(S, R)$-bimodules. The composition is induced by tensoring bimodules. Notice that there is a natural embedding $\text{Ass} \hookrightarrow \mathcal{B}\text{Ass}$. Denote by $\mathcal{M}\text{Ass}$ class of morphisms of the form $f \circ m$, where $m$ is the isomorphism class of an invertible bimodule (Morita equivalence) and $f$ belongs to $\text{Hom}(\text{Ass})$.

**C10.3.4. Proposition.**

(a) $\mathcal{M}\text{Ass}$ is a subcategory of $\mathcal{B}\text{Ass}$.

(b) The natural functor $\mathcal{B}\text{Ass}^o \to |\text{Cat}|^o$ is fully faithful.

(c) The restriction of the functor (b) to the subcategory $\mathcal{M}\text{Ass}^o \to (|\text{Cat}|^o/\text{SpZ})_{\text{aff}}$ induces an equivalence.

**Proof.** The assertion follows from C10.3.3 and 6.4.1.

**C10.4. Additive monads and continuous monads.** For any $X \in \text{Ob}|\text{Cat}|^o$ such that the category $C_X$ is (pre)additive, denote by $\text{Mon}_X^a$ the category of additive monads on $X$, i.e. monads $(F, \mu)$ on $C_X$ with an additive functor $F$. We denote by $\text{Mon}_X^c$ the category of continuous monads on $X$.

**C10.4.1. Proposition.** Let $R$ be an associative unital ring and $X = \text{Sp}(R)$. The inclusion functor $j^* : \text{Mon}_X^c \hookrightarrow \text{Mon}_X^a$ has a left adjoint.

**Proof.** Let $F = (F, \mu)$ be an additive monad on $C_{\text{Sp}(R)}$, and let $f : \text{Sp}(F/R) \to \text{Sp}(R)$ denote the canonical morphism with inverse image functor $V \mapsto (F(V), \mu(V))$. In
follows from (a) that the map $F$ is a ring morphism, morphisms $(G, \xi)$ which depends functorially on $F$ in a unique way (thanks to universal property of colimits).

Since every $F$ tors module, since every $j_x$ of continuous functors, preserves colimits of small diagrams. Since for any morphism is identical.

left adjoint to the inclusion functor $j_{\bullet}$ such that the adjunction morphism $\varphi_F (M)$ is an isomorphism for any projective $R$-module $M$ of finite type.

In fact, the isomorphism $F_c(R) \xrightarrow{\sim} F(R)$ extends to the functor $F_c = F(R) \otimes_R - : R - \text{mod} \rightarrow R - \text{mod}$. Thus we obtain a monad $\mathcal{F}_c$ on $R - \text{mod}$. The map $\mathcal{F} \rightarrow \mathcal{F}_c$ is functorial. We denote this functor by $j_{\bullet}$.

The latter means that the map $\varphi$ is a monad morphism. The function $\varphi : F \mapsto \varphi_F$ defines a functor morphism $j_{\bullet}^* \mapsto Id_{\text{Mon}^X_{\bullet}}$ which is an adjunction morphism. The other adjunction morphism, $Id_{\text{Mon}^X_{\bullet}} \mapsto j_{\bullet}^*$ is identical.

Let $C_X$ be an additive category. Let $End_a(C_X)$ denote the category of additive functors $C_X \rightarrow C_X$ and $End_c(C_X)$ its full subcategory formed by continuous endofunctors.

**C10.4.2. Proposition.** Let $C_X$ be an additive category with small colimits. Then

(a) The inclusion functor $End_c(C_X) \rightarrow End_a(C_X)$ has a left adjoint.

(b) The inclusion functor $j^*: \text{Mon}^X_{\bullet} \hookrightarrow \text{Mon}^X_{\bullet}$ has a left adjoint.

**Proof.** (a) Let $F$ be a functor $C_X \rightarrow C_X$. Consider the category $End_c(C_X)/F$ whose objects are pairs $(G, \xi)$, where $G$ is a continuous functor $C_X \rightarrow C_X$ and $\xi$ is a functor morphism, morphisms $(G, \xi) \mapsto (G', \xi')$ are functor morphisms $\psi: G \rightarrow G'$ such that $\xi = \xi' \circ \psi$. Let $D_F$ denote the diagram

$$End_c(C_X)/F \rightarrow End(C_X), \quad (G, \xi) \mapsto G.$$ 

We denote a colimit of the diagram $D_F$ by $F_c^*$. The functor $F_c^*$, being a colimit of a diagram of continuous functors, preserves colimits of small diagrams. Since for any $x, y \in Ob C_X$, there exist morphisms $F_c^*(x) \rightarrow y$, i.e. the category $F_c^*/y$ is non-empty, the assignment to every $y \in Ob C_X$ a colimit of the functor $F_c^*/y \rightarrow C_X$, $(x, F_c^*(x) \xrightarrow{\xi} y) \mapsto x$, determines a right adjoint, $F_c^*$, to $F_c^*$.

It follows from the construction that there is a canonical morphism $\epsilon_j(F): F_c^* \rightarrow F$ which depends functorially on $F$. If $F$ is continuous, then $\epsilon_j(F)$ can be chosen to be an identical morphism. This shows that $j^*: F \mapsto F_c^*$ is a left adjoint to the inclusion functor $j_{\bullet}: End_c(C_X) \hookrightarrow End_a(C_X)$ and $\epsilon_j$ is an adjunction morphism. The other adjunction morphism is identical.

(b) If $\mathcal{F} = (F, \mu)$ is a monad on $C_X$, then there is a unique monad structure, $\mu_c : (F_c^*)^2 \rightarrow F_c^*$ such that the adjunction morphism $F_c^* \rightarrow F$ is a monad morphism. It follows from (a) that the map $\mathcal{F} = (F, \mu) \mapsto \mathcal{F}_c = (F_c^*, \mu_c)$ extends to a functor which is left adjoint to the inclusion functor $j^*: \text{Mon}^X_{\bullet} \hookrightarrow \text{Mon}^X_{\bullet}$. Details are left to the reader. ■
C10.4.3. Proposition. Suppose the category $C_X$ has limits and colimits of small diagrams. Then the inclusion functor $j^* : \text{Mon}_C X \hookrightarrow \text{Mon}_X$ has a left adjoint.

Proof. Let $F = (F, \mu)$ be a monad on $X$. Since $\text{Id}_{C_X}$ is a continuous functor and there is a morphism $\eta : \text{Id}_{C_X} \to F$ (the unit of $F$), the category $\text{End}_c(C_X)/F$ (see the part (a) of the argument of C10.4.2) is non-empty. Let $D_F$ denote the standard functor $\text{End}_c(C_X)/F \to \text{End}(C_X)$. For any two objects, $x, y$ of the category $C_X$, we have isomorphisms:

$$C_X(F_c(x), y) \simeq \lim C_X(D_F(x), y) \simeq \lim C_X(x, D^\vee_F(y)).$$

Here $D^\vee_F$ denotes the diagram $\text{End}_c(C_X)/F \to \text{End}(C_X)$ which assigns to any object $(G^*, \xi)$ of $\text{End}_c(C_X)/F$ a right adjoint, $G_*$, to $G^*$ and to any morphism $(G^*, \xi) \to (H^*, \nu)$ the corresponding morphism $H_* \to G_*$ of right adjoint functors. If the category $C_X$ has limits, there exists a limit of $D^\vee_F(y)$. Choosing this limit for each $y$, we define a functor which is a right adjoint to $F_c$. ■

References.

[BeDr] A. Beilinson, V. Drinfeld, Hitchin’s integrable system, preprint, 1999
[BO] A. Bondall, D. Orlov, Semiorthogonal decompositions for algebraic varieties, alg-geom/9506012


[V1] A.B. Verevkin, On a noncommutative analogue of the category of coherent sheaves on a projective scheme, Amer. Math. Soc. Transl. (2) v. 151, 1992