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ABSTRACT

Accelerated life testing has been used for years in engineering. Test units are run at high stress and fail sooner than at design stress. The lifetime at design stress is estimated by extrapolation using a regression model. This paper considers the optimum design of accelerated life tests in which two levels of stresses, high and low are constantly applied. For the exponential model the expected value of an exponential loss function of the parameter is to be used. The initial sample proportion allocated to the high stress which minimizes the expected loss function is determined. In the agriculture context, plants or animal may be the items placed on test and dosage of a chemicals, amount of fertilizer, may be the stress variable. In this paper I suggest several potential applications of constant testing in agriculture and present inferential procedure in the case in which observations have the exponential distribution.

1. INTRODUCTION

This paper introduces constant-stress testing to the agricultural research community. The paper begins by reviewing the engineering origins of constant-stress testing. See Nelson (1990) and Khamis (1997). Accelerated life testing (ALT) quickly yields information on product life. Test units are run at high stress and fail sooner than at design stress. The lifetime at design stress is estimated by extrapolation using a regression model. In constant-stress testing, a test unit is run at a fixed stress until either failure occurs or the test is terminated. For instance, in replacing 30 lightbulbs at each of three voltages (130, 140,150), a constant stress design might place10 lightbulbs at each of three voltageas an optimum criterion to estimate the distribution of the lifetime of the lightbulb. For the exponential model the expected value of an exponential loss function of the parameter is to be minimized to obtain the optimum design. See (Higgins and Tsokos, 1980). In the agriculture context, plants or animal may be the items placed on test and dosage of a chemicals, amount of fertilizer, may be the stress variable. In this paper I suggests several potential applications of constant testing in agriculture and presents an inferential procedure in the case in which observations have the exponential distribution.

2. EXAMPLES IN AGRICULTURE

Optimum design for constant-stress testing has not been wildly used in agriculture. Such testing is potentially useful when a stress variable is applied to a plant, or other experimental unit. The following examples are included to demonstrate the applications.

Example 1. A researcher may be interested in modeling the effect of a potentially lethal drug on some physical characteristic of a laboratory animal. Here the stress variable would be drug level and the response would be the time it takes to reach some critical life stage (e.g. time to reach a critically low white blood count or time to death). In constant-stress testing, with 30 animals, 10 animals may be assigned to low stress and the remaining 20 to the high stress. When an animal reaches the desired critical life stage or death after having been administrated just a single level of the drug, the animal will be considered a failure.

Example 2. Similar to example one, a researcher may be interested in the time it takes a plant to reach a critical life stage or death after it has been deprived of an essential nutrient. The typical constant-stress experiment would place each plant under just one level of the nutrient. Some of the plants would subjected to a low nutrient level, and the remaining plants will receive the high level of the nutrient. The plant are observed until a critical life stage or death occurs.

Example 3. A veterinarian may be interested in determining the effects of exercise on the time it takes an animal to reach a certain physiological state. In constant-stress testing, each animal would be placed at single exercise level during a given run of experiment, and the time it takes to reach the desire state, if at all, would be recorded.

3. THE MODEL

In this section the optimum constant-stress test is derived for the exponential distribution. Although optimum constant-stress design has been investigated extensively, this result appears to give different than those previously published.

Basic Assumptions

The notation used in the following discussion are described as:

(1) Testing is done at stress levels x_1 and x_2 where $x_1 < x_2$.

(2) The distribution of the test unit is exponential. That is

$$f_1(t) = \left|\theta_1 \exp(-\theta_1 t)\right| , \qquad \text{(stress } x_1)$$

$$f2(t) = \theta_2 \exp\left(-\theta_2 t\right) , \quad t \geq 0 , \qquad \text{(stress } x_2).$$

where θ_i at stress x_i is assumed to be described by

$$\log(\theta_i) = \beta_o + \beta_1 x_i \tag{1}$$

(3) The lifetimes of test units are independent and identically distributed.

(4) All n units are placed on test, n_1 units assigned to the low stress x_1, and the remaining units $n_2=n-n_1$ are assigned to the high stress x_2. The test continue until all units fail.

4. THE ESTIMATION METHOD

The likelihood function from observations $T_{ij}=t_{ij}$, i=1,2, j=1,2...,n_i is:

$$L(\theta_1,\theta_2) = \prod_{j=1}^{n_1}\left[\theta_1 \exp(-\theta_1 t_{1j})\right]\prod_{j=1}^{n_2}\left[\theta_2 \exp(-\theta_2 t_{2j})\right] \tag{2}$$

where $n=n_1+n_2$. Substituting (1) for θ_i, i=1,2, in (2), we find the log likelihood function as a function of unknown parameters β_o and β_1. That is,

$$\log L(\beta_o,\beta_1) = n\,\beta_o + (n_1x_1+n_2x_2)\beta_1 - U_1\exp(\beta_o+\beta_1x_1)$$
$$- U_2\exp(\beta_o+\beta_1x_2)$$

where

$$U_1 \equiv \sum_{j=1}^{n_1} t_{1j} \quad \text{and} \quad U_2 \equiv \sum_{j=1}^{n_2} t_{2j}\ .$$

MLEs for the model parameters β_o and β_1 can be obtained explicitly by solving the following two equations

$$\frac{\partial \log L(\beta_o,\beta_1)}{\partial\beta_o} = -(n_1+n_2) + U_1\exp(-\beta_o-\beta_1x_1) + U_2\exp(-\beta_o-\beta_1x_2)$$
$$=0$$
$$\frac{\partial \log L(\beta_o,\beta_1)}{\partial\beta_1} = -(n_1x_1+n_2x_2) + U_1\exp(-\beta_o-\beta_1x_1)x_1 + U_2\exp(-\beta_o-\beta_1x_2)x_2$$
$$=0\ .$$

That is,

$$\hat{\beta}_0 = \frac{x_2 \ln\frac{n_1}{U_1} - x_1 \ln\frac{n_2}{U_2}}{x_2 - x_1}$$

$$\hat{\beta}_1 = \frac{\ln\left(\frac{n_2 U_1}{n_1 U_2}\right)}{x_2 - x_1} \tag{3}$$

Lemma: Suppose T_{ij} is distributed exponentially with scale parameter θ_i , i=1,2 , j=1,2,...,n_i. Then the random variables

$$2n_i e^{\beta_0+\beta_1 x_i} m_i \quad \text{and} \quad 2n_i e^{\beta_0+\beta_1 x_i}(t_i - n_i m_i)$$

where

$$m_i = \min\left\{ t_{ij} \quad j = 1, 2, \ldots, n_i \right\}$$

are independent and distributed chi-square with 2 and ($2n_i$ - 2) degrees of freedom respectively. See Lawless (1982).

From the above lemma we have

$$2n_i e^{\beta_0+\beta_1 x_i} t_{i.}$$

is distributed chi-square distribution with $2n_i$ degrees of freedom and the ratio

$$\frac{(2e^{\beta_0+\beta_1 x_1} t_{1.})/2n_1}{(2e^{\beta_0+\beta_1 x_2} t_{2.})/2n_2} = \frac{U_1\ n_2}{U_2\ n_1}\ e^{\beta_1\ (x_1 - x_2)} \tag{4}$$

is distributed F with $2n_1$ and $2n_2$ degrees of freedom . Using the expression in (3), (4) also becomes

$$\frac{U_1\ n_2}{U_2\ n_1} = e^{\hat{\beta}_1 (x_2 - x_1)} \quad . \tag{5}$$

Substitute (5) in (4) to get

$$e^{(\hat{\beta}_1-\beta_1)(x_2-x_1)}$$

which distributed F with $2n_1$, and $2n_2$, degrees of freedom.

An exponential loss function of the parameter β_1 for given values of x_1 and x_2 can be written as

$$L(\hat{\beta}_1,\beta_1) = \left(e^{(\hat{\beta}_1-\beta_1)(x_2-x_1)}-1\right)^2 \qquad (6)$$

Note that the loss function of β_1 is used because the rate of change of the log mean lifetime over stress levels is important in applications. In order to find the expected value of the loss function (6), it is important to note that the F distribution doesn't have a second moment when the second degrees of freedom is less than or equal to 4. Therefore designs are restricted to the case where $3 \le n_2 \le n-1$. Also, we note n_2 has a binomial distribution with n and π_2 parameters. That is

$$P(n_2=s|3\le n_2\le n-1)=\frac{C_k^n\ \pi_2^s(1-\pi_2)^{n-s}}{\sum_{s=3}^{n-1} C_k^n\ \pi_2^s(1-\pi_2)^{n-s}}$$

Thus, the expected value of the loss function for the parameter β_1 is

$$E\left((e^{(\hat{\beta}_1-\beta_1)(x_2-x_1)}-1)^2|3\le n_2\le n-1\right) =$$

$$\frac{\sum_{k=3}^{n-1}\left[\left(\frac{k}{k-1}\right)^2\frac{n-1}{(n-k)(k-2)}+\left(\frac{1}{k-1}\right)^2\right] C_k^n\ \pi_2^k(1-\pi_2)^{n-k}}{\sum_{k=3}^{n-1} C_k^n\ \pi_2^k(1-\pi_2)^{n-k}}\ . \qquad (7)$$

Note that this expression is independent of any stress level.

5. OPTIMUM TESTS DESIGN

A design is defined to be optimum if it minimize the expected value of the loss function. The design which minimizes (7) depends on the proportion π_2 of the test units allocated to the stress x_2 with the restriction $0<\pi_2<1$. The optimum design for π^*_2 which minimizes (7) can be found numerically. Khamis (1997) obtained for given values of x1=0.2 and x2=1.0 the optimum $\pi^*_2=0.25$ which also minimize the asymptotic variance of MLE of the log mean lifetime at the design stress for large n. This result is asymptotic. However, the optimum design here is exact and depends on the sample size. Table 1 shows the optimum π^*_2 that minimizes (7) for different sample sizes (n=10, 15, 20, 25, 30, 35, 40, 50) . Table 1 also shows the relative efficiency of the optimum design obtained by Khamis (1997). The relative efficiency , denoted Reff, is defined by

$$Reff = \frac{\min_{\pi_2} E\left((e^{(\hat{\beta}_1-\beta_1)(x_2-x_1)}-1)^2 | 3 \le n_2 \le n-1\right)}{\min_{\pi_2=0.25} E\left((e^{(\hat{\beta}_1-\beta_1)(x_2-x_1)}-1)^2 | 3 \le n_2 \le n-1\right)}$$

For example when the sample size is 15 the proportion of experimental units that fail at high stress x_2 under the optimal design is .62, and this design is 34% more efficient than the optimum design obtained by maximum likelihood method.

Table1: Comparing the efficiencies of the approximate and exact methods

n	10	15	20	25	30	35	40	50	80
$\hat{\pi}_2$	0.68	0.62	0.59	0.58	0.57	0.56	0.55	0.54	0.53
Reff	0.46	0.34	0.32	0.34	0.38	0.43	0.48	0.55	0.63

6. References

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