Conference on Applied Statistics in Agriculture 1998 - 10th Annual Conference Proceedings

OPTIMUM DESIGN ON STEP-STRESS LIFE TESTING

C. Xiong

OPTIMUM DESIGN ON STEP-STRESS LIFE TESTING

C. XIONG

Department of Mathematics, Southeast Missouri State University

Abstract This paper presents exact optimum test plans for simple time-step stress models in accelerated life testing. An exponential life distribution with a mean that is a log-linear function of stress, and a cumulative exposure model are assumed. Maximum likelihood methods are used to estimate the parameters of such models. Optimum test plans are obtained by minimizing the mean square error between the maximum likelihood estimate of a certain moment of the lifetime at a design stress and the real moment. The advantage of our optimum test plans is that it does not require large number of items to be tested. We also compare our results with test plans obtained by minimizing the asymptotic variance of the maximum likelihood estimate of the mean life at a design stress.

Keywords Cumulative exposure model; exponential distribution; extrapolation; loss function; maximum likelihood

1. INTRODUCTION

Accelerated life testing of a product or material is used to quickly obtain information on its life distribution. Test units are tested at high-than-normal levels of stress such as high temperature, voltage, pressure, vibration, cycling rate, or load to induce early failure. Data obtained from accelerated life testing are then analyzed based on models which relate the life time to stress. Then the method of extrapolation is used to estimate the life distribution at a design stress.

Accelerated life testing can be carried out using either constant stress or step-stress. The time-step stress scheme applies stress to the experimental units in the way that the stress setting of a unit will be changed at prespecified times. Generally, a test unit starts at a specified low stress. If the unit does not fail at a specified time, stress on it is raised and held a specified time. Stress is repeatedly increased and held, until the test unit fails. A simple time-step stress accelerated life testing plan uses only two stress levels. The problem of making inferences and finding optimum test plans in accelerated life testing has been studied by many authors. Meeker and Nelson (1975) obtained optimum test plans for Weibull and extreme value distributions with censored data. Nelson and Kielpinski (1976) studied optimum test plans for normal and lognormal life distributions. Nelson (1980) obtained maximum likelihood estimators for the parameters of a Weibull distribution under the inverse power law using the breakdown time data of an electrical insulation. Miller and Nelson (1983) studied optimum test plans which minimized the asymptotic variance of the maximum likelihood estimator of the mean life at a design stress for simple step-stress testing when all units were run to failure. Bai, Kim and Lee (1989) further studied the similar optimum simple step-stress accelerated life tests for the case where a prespecified censoring time was involved. Meeker and Escobar (1993) briefly surveyed optimum test plans in accelerated life testing. Nelson (1982, 1990) provided an extensive and comprehensive source for theory and examples for accelerated testing.

While most of the above mentioned work obtained optimum test plans by minimizing the asymptotic variance of the maximum likelihood estimate of the mean life at a design stress, this paper considers the exact optimum test plan for the simple time-step stress tests with exponential life distributions at constant stresses and the cumulative exposure model. The mean life at a constant stress level is assumed to be a log-linear function of the stress. Our criterion of optimum test plans is to minimize the mean square error between the maximum likelihood estimate of a certain moment of the lifetime at a design stress and the real moment. We also present some numerical results to compare our test plans with the test plan of Miller and Nelson (1983) obtained by minimizing the asymptotic variance of the maximum likelihood estimate of the mean life at a design stress. The data of Miller and Nelson (1983) are also used to illustrate our test plans.

Notations

x_0	design stress
x_1, x_2	design stress
n	number of test units
τ	stress change point
n_1	number of units failed before stress change
n_2	number of units survived the stress change point
T_{ij}	failure time of j-th test unit under stress $x_i, i = 1, 2, j = 1, 2, ..., n_i$
$T_{i\cdot}$	$\sum_{j=1}^{n_i} T_{ij}, i = 1, 2.$
θ_i	mean life at stress $x_i, i = 0, 1, 2$
$F_i(.)$	cumulative distribution function of exponential distribution with mean θ_i
$G(.)$	cumulative distribution function of a test unit under simple time-step stress test

Assumptions

1. Two test stress levels x_1 and x_2 are used with $x_1 < x_2$.
2. For any level of stress, the life distribution of a test unit is exponential.
3. At stress level x, the mean life of a test unit is a log-linear function of stress. That is,

$$\log \theta(x) = \alpha + \beta x, \tag{1}$$

where α and β are unknown parameters depending on the nature of the product and the method of test.

4. A cumulative exposure model holds. That is, the remaining life time of a test unit depends only on the cumulative exposure it has seen. (Miller and Nelson (1983))

2. MODEL AND MAXIMUM LIKELIHOOD ESTIMATION

Suppose that n test units are initially placed on low stress level x_1 and run until time τ, when stress is changed to x_2 and the test is continued until all units fail. n_1 failure times $\{T_{1j}\}_{j=1}^{n_1}$ are observed under stress x_1 and n_2 failure times $\{T_{2j}\}_{j=1}^{n_2}$ are observed under stress x_2 after time τ. The assumptions of cumulative exposure model and exponentially distributed life at any constant stress imply that, the cumulative distribution function of a test unit under simple step-stress test is

$$G(t) = \begin{cases} F_1(t), & \text{for} \quad 0 \leq t < \tau \\ F_2(s + t - \tau), & \text{for} \quad \tau \leq t < \infty \end{cases}$$

where s is the solution of $F_2(s) = F_1(\tau)$.

Since $F_i(t) = 1 - e^{-t/\theta_i}$, $s = \theta_2\tau/\theta_1$. Thus, the probability density function of a test unit is

$$f(t) = \begin{cases} e^{-t/\theta_1}/\theta_1 & \text{for} \quad 0 \leq t < \tau \\ e^{-(t-\tau)/\theta_2 - \tau/\theta_1}/\theta_2 & \text{for} \quad \tau \leq t < \infty \end{cases} . \qquad (2)$$

The likelihood function from observations $T_{ij}, i = 1, 2, j = 1, 2, ..., n_i$, is then

$$L(\theta_1, \theta_2) = \prod_{j=1}^{n_1}[(1/\theta_1)\exp(-T_{1j}/\theta_1)]\prod_{j=1}^{n_2}[(1/\theta_2)\exp(-(T_{2j} - \tau)/\theta_2 - \tau/\theta_1)],$$

where $n_1 + n_2 = n$. Substituting (1) for θ_1 and θ_2 in the likelihood function, the log likelihood function is a function of unknown parameter α and β:

$$\log L(\alpha, \beta) = -n\alpha - (n_1x_1 + n_2x_2)\beta - U_1\exp(-\alpha - \beta x_1) - U_2\exp(-\alpha - \beta x_2),$$

$$U_1 = T_1. + n_2\tau$$

$$U_2 = T_2. - n_2\tau.$$

Letting $\partial \log L(\alpha,\beta)/\partial\alpha = 0$ and $\partial \log L(\alpha,\beta)/\partial\beta = 0$ yields the maximum likelihood estimators for α and β when $n_1 > 0$ and $n_2 > 0$:

$$\widehat{\alpha} = (x_1 \log(n_2/U_2) - x_2 \log(n_1/U_1))/(x_2 - x_1)$$

$$\widehat{\beta} = (\log(n_1U_2/(n_2U_1)))/(x_2 - x_1).$$

3. OPTIMUM TEST PLANS

Suppose that n test units are tested according to model (2). We will only focus on the designs with $n > n_2 \geq 2$ (or equivalently, $1 \leq n_1 \leq n-2$). Let $\xi = (x_1 - x_0)/(x_2 - x_1)$ be the amount of extrapolation. Let $p = 1 - \exp(-\tau/\theta_1)$ be the probability that a test unit fails before the stress change time τ according to model (2). For $1 \leq k \leq n-2$, we define several notations:

$g_1(k,\xi,n) = 2^{-\frac{2\xi}{1+\xi}}\Gamma(n-k-\frac{2\xi}{1+\xi})/\Gamma(n-k);$

$g_2(k,\xi,n) = 2^{-\frac{\xi}{1+\xi}}\Gamma(n-k-\frac{\xi}{1+\xi})/\Gamma(n-k);$

$h_1(\theta_1,\tau,k,n) = k(2\theta_1^2 - \tau(\tau+2\theta_1)(1-p)/p) + k(k-1)(\theta_1 - \tau(1-p)/p)^2$

$+2k(n-k)\tau(\theta_1 - \tau(1-p)/p) + ((n-k)\tau)^2;$

$h_2(\theta_1,\tau,k,n) = k(\theta_1 - \tau(1-p)/p) + (n-k)\tau.$

Let $\widehat{\theta_0} = \exp(\widehat{\alpha} + \widehat{\beta}x_0)$ be the maximum likelihood estimate of the mean life $\theta_0 = \exp(\alpha + \beta x_0)$ at design stress x_0. In order to measure the distance between $\widehat{\theta_0}$ and θ_0, Miller and Nelson (1983) used the square loss function $(\widehat{\theta_0} - \theta_0)^2$ and obtained the optimum test plans by minimizing the asymptotic expectation of the loss. We propose to use the loss function $((\widehat{\theta_0}/\theta_0)^{1/(1+\xi)} - 1)^2$ to measure the distance between $\widehat{\theta_0}$ and θ_0. This loss function has the similar mathematical property as the square loss function $(\widehat{\theta_0} - \theta_0)^2$. More specifically, if the maximum likelihood estimate $\widehat{\theta_0}$ of

θ_0 at design stress x_0 is close to the real θ_0 at design stress x_0, then $\widehat{\theta_0}/\theta_0$ would be close to 1, and $((\widehat{\theta_0}/\theta_0)^{1/(1+\xi)}-1)^2$ would be close to zero. Our criterion of optimum test plans is to minimize the expectation of the loss $((\widehat{\theta_0}/\theta_0)^{1/(1+\xi)}-1)^2$. Notice that $\widehat{\theta_0}^{1/(1+\xi)}\Gamma(1+1/(1+\xi))$ is the maximum likelihood estimate of the $1/(1+\xi)-$th moment $\theta_0^{1/(1+\xi)}\Gamma(1+1/(1+\xi))$ of the lifetime at design stress x_0, $((\widehat{\theta_0}/\theta_0)^{1/(1+\xi)}-1)^2$ is a multiple of the square error loss between the maximum likelihood estimate of the $1/(1+\xi)-$th moment of the lifetime and the real $1/(1+\xi)-$th moment at design stress x_0. The expected loss, given $1 \le n_1 \le n-2$, can be computed as (see Appendix for the derivation)

$$E(((\widehat{\theta_0}/\theta_0)^{1/(1+\xi)}-1)^2|1 \le n_1 \le n-2)$$
$$= 1+\textstyle\sum_{k=1}^{n-2}[\theta_1^{-2}(2n-2k)^{2\xi/(1+\xi)}h_1(\theta_1,\tau,k,n)g_1(k,\xi,n)k^{-2}-2\theta_1^{-1}(2n-2k)^{\xi/(1+\xi)}$$
$$h_2(\theta_1,\tau,k,n)g_2(k,\xi,n)k^{-1}]\tbinom{n}{k}p^k(1-p)^{n-k}/(1-(1-p)^n-np^{n-1}(1-p)-p^n). \qquad (3)$$

To find the optimum test plan, we need to minimize $E(((\widehat{\theta_0}/\theta_0)^{1/(1+\xi)}-1)^2|1 \le n_1 \le n-2)$ over the choices of τ, x_1 and x_2. Miller and Nelson (1983) pointed out that x_1 (x_2) should be chosen as low (high) as possible as long as the choices do not cause failure modes different from those at the design stress so that the model remains valid over the range of the test and design stresses. We will assume that x_1 and x_2 are specified by experimenters. Our optimization criterion is then to minimize $E(((\widehat{\theta_0}/\theta_0)^{1/(1+\xi)}-1)^2|1 \le n_1 \le n-2)$ over τ. The optimum stress change time τ can be found by solving the equation

$$\frac{\partial E(((\widehat{\theta_0}/\theta_0)^{1/(1+\xi)}-1)^2|1\le n_1\le n-2)}{\partial \tau} = 0. \qquad (4)$$

There exists no close form solution to equation (4) in general, and hence the equation has to be solved by numerical methods such as Newton-Raphson's method. Since unknown parameter θ_1 is involved in $E(((\widehat{\theta_0}/\theta_0)^{1/(1+\xi)}-1)^2|1 \le n_1 \le n-2)$, it has to be estimated from experience, similar data or preliminary tests before an optimum test plan can be found.

Miller and Nelson (1983) use model (2) to obtain the optimum stress change time τ by minimizing the asymptotic variance of the maximum likelihood estimate of the mean life at design stress x_0. Notice that our results use a different criterion for the optimization of τ and provides the exact optimum test plans. Table 1 presents a comparison between the optimum stress change time τ^* of Miller and Nelson (1983) and our optimum stress change time τ^{**} for several different choices of ξ. We choose $\theta_1 = 10$ and $n = 30$ in Table 1. Notice that, as $\xi \to 0$, $((\widehat{\theta_0}/\theta_0)^{1/(1+\xi)} - 1)^2$ is approximately a multiple of the square error loss between $\widehat{\theta_0}$ and θ_0. This explains why the results of Miller and Nelson (1983) and our results become very close when ξ is small in Table 1.

Table 1. Comparison between optimum asymptotic test plan τ^* and optimum exact test plan τ^{**}

ξ	τ^{**}	τ^*
3.00	8.96	8.47
2.50	9.21	8.75
2.00	9.58	9.16
1.75	9.84	9.45
1.50	10.16	9.81
1.25	10.60	10.30
1.00	11.23	10.99
0.75	12.18	12.04
0.50	13.88	13.86
0.25	17.84	17.91

To examine the effect of the sample size n on the optimum stress change time τ, we compute the optimum stress change time τ when $x_0 = 0, x_1 = 1, x_2 = 2$, $\theta_1 = 10$ and $n = 10$, 20, 30, 50, 80, 100, 150, 200. We find that, after n reaches 80, the optimum stress change time stabilizes at about $\tau^{**} = 11.12$. Finally, in order to compute the optimum stress change time τ^{**}, one must know θ_1 in advance. Suppose one incorrectly uses θ_1' for θ_1. Then the actual test plan is no longer optimum and has a higher expected loss. Table 2 presents the percentage of the increase of the expected loss at the optimum stress change time $\tau^{**} = 11.23$, $(E(((\widehat{\theta}_0/\theta_0')^{1/(1+\xi)} - 1)^2 | 1 \leq n_1 \leq n-2) - E(((\widehat{\theta}_0/\theta_0)^{1/(1+\xi)} - 1)^2 | 1 \leq n_1 \leq n-2))/E(((\widehat{\theta}_0/\theta_0)^{1/(1+\xi)} - 1)^2 | 1 \leq n_1 \leq n-2)$, when $\xi = 1, \theta_1 = 10$, $n = 30$, and θ_1 is misspecified as θ_1'.

Table 2. The effect of misspecified θ_1

θ_1'/θ_1	% of the increase of the expected loss
2.00	51.62%
1.75	32.34%
1.50	16.79%
1.25	5.26%
1.00	0.00%
0.75	14.33%
0.50	166.77%
0.25	1015.36%

Example:

Miller and Nelson (1983) reported an accelerated life test with 76 times (in minutes) to breakdown of an insulating fluid at constant voltage stresses (kV). The extreme (transformed) test stresses are $x_1 = \ln(26.0) = 3.2581$, and $x_2 = \ln(38.0) = 3.6376$. The (transformed) design stress is $x_0 = \ln(20.0) = 2.9957$. Miller and Nelson (1983) obtained the maximum likelihood estimates of the model parameters for those data: $\widehat{\alpha} = 64.912$ and $\widehat{\beta} = -17.704$. The maximum likelihood estimates of the means at stresses x_1 and x_2 are $\widehat{\theta_1} = 1380$ minutes and $\widehat{\theta_2} = 1.67$ minutes. The estimate of the mean life at the design stress is $\widehat{\theta_0}$ =144,000 minutes. By minimizing the asymptotic variance of the maximum likelihood estimate of the mean at the design stress, Miller and Nelson (1983) also reported the optimum stress change point $\tau^* = 1707$ minutes. By minimizing $E(((\widehat{\theta}_0/\theta_0)^{1/(1+\xi)} - 1)^2 | 1 \le n_1 \le n-2)$, we found that the optimum stress change time is $\tau^{**} = 1729$ minutes when $n = 76$.

APPENDIX

We give the derivation for (3). First, Let a random variable T be distributed as in (2). Then it is easy to verify that the random variable

$$S = \begin{cases} T/\theta_1 & \text{for} \quad 0 \le T < \tau \\ (T-\tau)/\theta_2 + \tau/\theta_1 & \text{for} \quad \tau \le T < \infty \end{cases} \tag{5}$$

is exponentially distributed with mean 1. Thus, for any constant $a > 0, (S-a)|S > a$ is also exponentially distributed with mean 1.

The following lemma from Lawless (1982) is used in our derivation.

Lemma: Suppose that $\{S_i\}_{i=1}^n$ are i.i.d. exponential random variables with mean 1. Let $M = \min\{S_i, i = 1, 2, ..., n\}$ and $S. = \sum_{i=1}^n S_i$. Then $2nM$ has a χ^2-distribution with 2 degrees of freedom and $2(S. - nM)$ has a χ^2-distribution with $2n-2$ degrees of freedom. Further, these two random variables are independent.

Next we transfer all random variables T_{ij} into S_{ij} through (5). Let $1 \le k \le n-2$. Given $n_1 = k$

(or equivalently, given $n_2 = n-k \geq 2$), by the Lemma, $2U_2/\theta_2$ has a χ^2−distribution with $2(n-k)$ degrees of freedom. Thus

$$g_1(k,\xi,n) = E((2U_2/\theta_2)^{-2\xi/(1+\xi)}|n_1 = k)$$

$$= 2^{-2\xi/(1+\xi)}\Gamma(n-k-2\xi/(1+\xi))/\Gamma(n-k)$$

and

$$g_2(k,\xi,n) = E((2U_2/\theta_2)^{-\xi/(1+\xi)}|n_1 = k)$$

$$= 2^{-\xi/(1+\xi)}\Gamma(n-k-\xi/(1+\xi))/\Gamma(n-k).$$

Since the distribution of $T_{1\cdot}$, given $n_1 = k$, is the same as the distribution of $\sum_{j=1}^{k} T_{1j}$, given $T_{11} \leq \tau, T_{12} \leq \tau, ..., T_{1k} \leq \tau$. It follows that

$$E(T_{1\cdot}|n_1 = k) = kE(T|T \leq \tau)$$

$$= k(\theta_1 - \tau(1-p)/p)$$

and

$$E(T_{1\cdot}^2|n_1 = k) = kE(T^2|T \leq \tau) + k(k-1)(E(T|T \leq \tau))^2$$

$$= k(2\theta_1^2 - \tau(\tau + 2\theta_1)(1-p)/p) + k(k-1)(\theta_1 - \tau(1-p)/p)^2$$

where $p = P(T \leq \tau) = 1 - \exp(-\tau/\theta_1)$. Hence

$$h_1(\theta_1,\tau,k,n) = E(U_1^2|n_1 = k)$$

$$= k(2\theta_1^2 - \tau(\tau + 2\theta_1)(1-p)/p) + k(k-1)(\theta_1 - \tau(1-p)/p)^2$$

$$+2k(n-k)\tau(\theta_1 - \tau(1-p)/p) + ((n-k)\tau)^2$$

and

$$h_2(\theta_1,\tau,k,n) = E(U_1|n_1 = k)$$

$$= k(\theta_1 - \tau(1-p)/p) + (n-k)\tau.$$

Since

$$\widehat{\theta_0}/\theta_0 = (\theta_1^{-1}U_1)^{1+\xi}n_2^{\xi}/((\theta_2^{-1}U_2)^{\xi}n_1^{1+\xi}),$$

$$E(((\widehat{\theta_0}/\theta_0)^{1/(1+\xi)} - 1)^2 | 1 \leq n_1 \leq n-2)$$
$$= E((\theta_1^{-1}U_1)^2 n_2^{2\xi/(1+\xi)} / ((\theta_2^{-1}U_2)^{2\xi/(1+\xi)} n_1^2) | 1 \leq n_1 \leq n-2)$$
$$-2E((\theta_1^{-1}U_1) n_2^{\xi/(1+\xi)} / ((\theta_2^{-1}U_2)^{\xi/(1+\xi)} n_1) | 1 \leq n_1 \leq n-2) + 1. \quad (6)$$

Finally, since U_1 and U_2, given $n_1 = k$, are independent,

$$E((\theta_1^{-1}U_1)^2 n_2^{2\xi/(1+\xi)} / ((\theta_2^{-1}U_2)^{2\xi/(1+\xi)} n_1^2) | 1 \leq n_1 \leq n-2)$$
$$= \sum_{k=1}^{n-2} \theta_1^{-2}(2n-2k)^{2\xi/(1+\xi)} h_1(\theta_1, \tau, k, n) g_1(k, \xi, n) k^{-2} \binom{n}{k} (1 - \exp(-\tau/\theta_1))^k$$
$$\times (\exp(-\tau/\theta_1))^{n-k} / (1 - (1-p)^n - np^{n-1}(1-p) - p^n),$$
$$E((\theta_1^{-1}U_1) n_2^{\xi/(1+\xi)} / ((\theta_2^{-1}U_2)^{\xi/(1+\xi)} n_1) | 1 \leq n_1 \leq n-2)$$
$$= \sum_{k=1}^{n-2} \theta_1^{-1}(2n-2k)^{\xi/(1+\xi)} h_2(\theta_1, \tau, k, n) g_2(k, \xi, n) k^{-1} \binom{n}{k} (1 - \exp(-\tau/\theta_1))^k$$
$$\times (\exp(-\tau/\theta_1))^{n-k} / (1 - (1-p)^n - np^{n-1}(1-p) - p^n).$$

(3) is now proved by combining the above equations in (6).

ACKNOWLEDGEMENTS

The author wishes to thank Dr. Paul I. Nelson and Dr. James J. Higgins, Department of Statistics, Kansas State University, for a careful reading of the manuscript.

REFERENCES

D. S. Bai, M. S. Kim and S. H. Lee, "Optimum Simple Step-Stress Accelerated Life Tests with Censoring," *IEEE Trans. Reliability* vol. 38, 5, pp. 528-532, 1989.

J. F. Lawless, *Statistical Models and Methods for Lifetime Data,* Wiley: New York, 1982.

W. Q. Meeker and L. A. Escobar, "A Review of Recent Research and Current Issues in Accelerated Testing," *International Statistical Review* vol. 61, 1, pp. 147-168, 1993.

W. Q. Meeker and W. B. Nelson, "Optimum Accelerated Life Tests for Weibull and Extreme Value Distributions and Censored Data," *IEEE Trans. Reliability* vol. R-24, pp. 321-332, 1975.

R. W. Miller and W. B. Nelson, "Optimum Simple Step-Stress Plans for Accelerated Life Testing," *IEEE Trans. Reliability* vol. R-32, 1, pp. 59-65, 1983.

W. B. Nelson, "Accelerated Life Testing — Step-Stress Models and Data Analysis," *IEEE Trans. Reliability* vol. R-29, 2, pp. 103-108, 1980.

W. B. Nelson, *Accelerated Life Testing: Statistical Models, Test Plans, and Data Analysis,* Wiley: New York, 1990.

W. B. Nelson, *Applied Life Data Analysis,* John Wiley: New York, 1982.

W. B. Nelson and T. J. Kielpinski, "Theory for Optimum Censored Accelerated Life Tests for Normal and Lognormal Life Distributions," *Technometrics* vol. 18, 1, pp. 105-114, 1976.