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CONFIDENCE INTERVALS FOR VARIANCE COMPONENTS IN ONE-WAY UNBALANCED DESIGNS

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Confidence Intervals for Variance Components in One-way Unbalanced Designs

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Abstract

Consider the one way unbalanced components of variance model given by

$$Y_{ij} = \mu + A_i + E_{ij},$$

$(i = 1, ..., a,\ j = 1, ..., b_i)$ where μ is an unknown constant parameter, A_i and E_{ij} are independent normal random variables with zero means and variances σ_A^2 and σ_E^2 respectively.
The problem is to obtain a confidence interval for σ_A^2 with confidence coefficient greater than or equal to a specified $1 - \alpha$. Three new procedures for obtaining confidence intervals for σ_A^2 are examined. These new methods are derived using unweighted means. These three methods are compared with a "standard" procedure based on confidence coefficients and expected "widths".

1 Introduction

In a one-way random effects model it is often of interest to find confidence intervals for the variance component σ_A^2. As an example suppose we are interested in the nitrogen content of the foliage in a large orchard. The two major sources of variation are the variance of nitrogen content for the leaves on a given tree (σ_E^2) and the variance among the nitrogen contents of the trees in the orchard (σ_A^2). In order to
measure the nitrogen content, a random sample of trees from the orchard is collected and a random sample of leaves is taken from each tree; Y_{ij} is the observed nitrogen content for the jth leaf from the ith tree sampled, and the

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model is a one-way random effects model $Y_{ij} = \mu + A_i + E_{ij}$. By observing Y_{ij} we want to find a confidence interval estimate for σ_A^2 with confidence coefficient $1-\alpha$.

No method of obtaining exact confidence intervals for σ_A^2 has been given, but five approximate methods will be discussed here. Three of them give confidence coefficients very close to $1-\alpha$. One of these methods is the Tukey-Williams procedure and was developed independently by Tukey (1951) and Williams (1962). Another was developed independently by Moriguti (1954) and Bulmer (1957). The third was developed by Howe (1974). These three methods have confidence coefficients close to $1-\alpha$ and it has been proved by Wang (1990) that the confidence coefficient for the Tukey-Williams procedure is $\geq 1-\alpha$. Two other methods labeled method 4 and method 5 which are derived using Bonferroni's method have confidence coefficients $\geq 1-2\alpha$. These five methods use the among sums of squares $= \sum_i \sum_j (\bar{Y_{i.}} - \bar{Y_{..}})^2$ and the within sums of square$= \sum_i \sum_j (Y_{ij} - \bar{Y_{i.}})^2$ which are scaled chi-squared and are independent.

For the unbalanced case however the among sums of squares are no longer scaled chi-squared and hence a problem arises. Burdick and Graybill (1984) gave an approximate method for obtaining confidence intervals for σ_A^2 for the unbalanced case but this method does not always have a confidence coefficient greater than the specified $1-\alpha$.

In this article three new methods A, B and C are proposed for finding confidence intervals for σ_A^2 for the unbalanced one-way design. At least one method, method A, has confidence coefficient $\geq 1-\alpha$.

2 The Balanced One-Way Classification

Consider the one-way random effects model

$$Y_{ij} = \mu + A_i + E_{ij} \qquad i = 1, ..., a; \quad j = 1, ..., b \tag{1}$$

where μ is a constant parameter, A_i and E_{ij} are independent normal random variables with zero means and variances σ_A^2 and σ_E^2 respectively. An ANOVA table is

Source of Variation (SV)	Mean Squares (MS)	Degrees of Freedom (DF)	Expected Mean Square (EMS)
Factor A	S_A^2	n_1	$b\sigma_A^2 + \sigma_E^2$
Error	S_E^2	n_2	σ_E^2

where

$$S_A^2 = \sum_{j=1}^{b} \sum_{i=1}^{a} (\bar{Y}_{i.} - \bar{Y}_{..})^2 / n_1,$$
$$S_E^2 = \sum_{1=1}^{a} \sum_{j=1}^{b_i} (Y_{ij} - \bar{Y}_{i.})^2 / n_2,$$

where $n_1 = a - 1$, $n_2 = a(b-1)$, $\bar{Y}_{i.} = \sum_{j=1}^{b} Y_{ij}/b$ and $\bar{Y}_{..} = \sum_{j=1}^{b} \sum_{i=1}^{a} Y_{ij}/ab$. The random variables $\bar{Y}_{..}$, S_A^2 and S_E^2 are complete sufficient statistics for this model, and $n_1 S_A^2 / E(S_A^2)$ and $n_2 S_E^2 / E(S_E^2)$ are independent chi-squared random variables with n_1 and n_2 degrees of freedom respectively.
For the balanced model in (1), we will display the five different methods referred to above for obtaining confidence intervals for σ_A^2.

Method 1: Tukey Williams (TW) Procedure

A $1 - \alpha$ lower and upper confidence limit for σ_A^2 given by the TW procedure are L_{TW} and U_{TW}, where

$$L_{TW} = [S_A^2 - (F_{1-\alpha:n_1,n_2} S_E^2)] / bF_{1-\alpha:n_1,\infty} \tag{2}$$

and

$$U_{TW} = [S_A^2 - (F_{\alpha:n_1,n_2} S_E^2)] / bF_{\alpha:n_1,\infty}. \tag{3}$$

Wang (1990) showed that $P[L_{TW} \leq \sigma_A^2 \leq U_{TW}] \geq 1 - \alpha$

Method 2: Howe (H) Procedure

The lower and upper $1 - \alpha$ confidence bound for σ_A^2 given by Howe is L_H and U_H respectively where

$$L_H = (1/b)[S_A^2 - S_E^2 - \sqrt{[(1 - F_{1-\alpha:n_1,\infty}^{-1})^2 (S_A^2)^2] + B(S_E^2)^2}] \qquad \text{if} \quad F \geq F_{1-\alpha:n_1,n_2}$$

$$L_H = 0 \qquad \text{if} \quad F < F_{1-\alpha:n_1,n_2}$$

(4)

$$U_H = (1/b)[S_A^2 - S_E^2 + \sqrt{[(1 - F_{\alpha:n_1,\infty}^{-1})^2 (S_A^2)^2] + A(S_E^2)^2}] \qquad \text{if} \quad F \geq F_{\alpha:n_1,n_2}$$

$$U_H = 0 \qquad \text{if} \quad F < F_{\alpha:n_1,n_2}$$

(5)

where $F = (S_A^2/S_E^2)$, $B = (1 - F^2_{1-\alpha:n_1,n_2}) - F^2_{1-\alpha:n_1,n_2}(1 - F^{-1}_{1-\alpha:n_1,\infty})^2$, and $A = (1 - F^2_{\alpha:n_1,n_2}) - F^2_{\alpha:n_1,n_2}(1 - F^{-1}_{\alpha:n_1,\infty})^2$.
For the Howe procedure it is not known if $P[L_H \leq \sigma_A^2 \leq U_H]$ is $\geq 1 - \alpha$.

Method 3: Bulmer-Moriguti (BM) Procedure

The lower and upper $1 - \alpha$ confidence bound for σ_A^2 using Bulmer-Morigutti's method is L_{BM} and U_{BM} where

$$\begin{aligned} L_{BM} &= (1/b)\{S_E^2[F^{-1}_{1-\alpha:n_1,\infty}F - 1 - F_{1-\alpha:n_1,n_2}F^{-1}(F_{1-\alpha:n_1,n_2}F^{-1}_{1-\alpha:n_1,\infty} - 1)]\} && \text{if } F \geq F_{1-\alpha:n_1,n_2} \\ L_{BM} &= 0 && \text{if } F < F_{1-\alpha:n_1,n_2} \end{aligned} \tag{6}$$

$$\begin{aligned} U_{BM} &= (1/b)\{S_E^2[F^{-1}_{\alpha:n_1,\infty}F - 1 - F_{\alpha:n_1,n_2}F^{-1}(F_{\alpha:n_1,n_2}F^{-1}_{\alpha:n_1,\infty} - 1)]\} && \text{if } F \geq F_{\alpha:n_1,n_2} \\ U_{BM} &= 0 && \text{if } F < F_{\alpha:n_1,n_2} \end{aligned} \tag{7}$$

where $F = S_A^2/S_E^2$.
For the Bulmer-Moriguti procedure it is not known if $P[L_{BM} \leq \sigma_A^2 \leq U_{BM}]$ is $\geq 1 - \alpha$

Two Other Methods

Methods 4 and 5 for obtaining confidence intervals, for σ_A^2 will be based on the $1 - \alpha$ confidence intervals for $\sigma_A^2 + \sigma_E^2/b$, σ_A^2/σ_E^2 and σ_E^2 respectively given below in (8), (9), (10). See Graybill (1976).

$$L_1 \leq \sigma_A^2 + \sigma_E^2/b \leq U_1 \tag{8}$$

where

$$\begin{aligned} L_1 &= S_A^2/bF_{1-\alpha/2:n_1,\infty} \\ U_1 &= S_A^2/bF_{\alpha/2:n_1,\infty}. \end{aligned}$$

$$L_2 \leq \sigma_A^2/\sigma_E^2 \leq U_2 \tag{9}$$

where

$$\begin{aligned} L_2 &= [(S_A^2/S_E^2F_{1-\alpha/2:n_1,n_2}) - 1]/b \\ U_2 &= [(S_A^2/S_E^2F_{\alpha/2:n_1,n_2}) - 1]/b. \end{aligned}$$

$$L_3 \leq \sigma_E^2 \leq U_3 \tag{10}$$

where

$$\begin{aligned} L_3 &= S_E^2/F_{1-\alpha/2:n_2,\infty} \\ U_3 &= S_E^2/F_{\alpha/2:n_2,\infty}. \end{aligned}$$

Method 4:

By the Bonferroni method the intersection of (8) and (10) gives the upper and lower confidence bounds L_4 and U_4 respectively for σ_A^2 with confidence coefficient $\geq 1-2\alpha$, where L_4 and U_4 are given by

$L_4 = L_1 - (U_3/b)$ and $U_4 = U_1 - (L_3/b)$.

Hence substituting for U_1, U_3, L_1, L_3 we get

$$P[L_4 \leq \sigma_A^2 \leq U_4] \geq 1-2\alpha$$

where

$$L_4 = S_A^2/bF_{1-\alpha:n_1,\infty} - S_E^2/bF_{\alpha:n_2,\infty} \tag{11}$$

and

$$U_4 = S_A^2/bF_{\alpha:n_1,\infty} - S_E^2/bF_{1-\alpha:n_2,\infty} \tag{12}$$

For this method $P[L_4 \leq \sigma_A^2 \leq U_4]$ is not always $\geq 1-\alpha$

Method 5:

By the Bonferroni method the intersection of (9) and (10) gives an upper and lower confidence limits L_5 and U_5 respectively for σ_A^2 with confidence coefficient $\geq 1-2\alpha$, where L_5 and U_5 are given by

$L_5 = (L_2)(L_3)$ and $U_5 = (U_2)(U_3)$.

Substituting for U_2, U_3, L_2, L_3 we get

$$P[L_5 \leq \sigma_A^2 \leq U_5] \geq 1-2\alpha$$

where

$$L_5 = S_A^2/(bF_{1-\alpha:n_1,\infty}F_{1-\alpha:n_1,n_2}) - S_E^2/(bF_{1-\alpha:n_2,\infty}) \quad (13)$$
$$U_5 = S_A^2/(bF_{\alpha:n_1,\infty}F_{\alpha:n_1,n_2}) - S_E^2/(bF_{\alpha:n_2,\infty}). \quad (14)$$

For this method $P[L_5 \leq \sigma_A^2 \leq U_5]$ is not always $\geq 1-\alpha$.

3 Unbalanced One-Way Design

The above five methods are appropriate for balanced one-way models. Now consider the unbalanced model given by

$$Y_{ij} = \mu + A_i + E_{ij} \qquad i = 1, ..., a \quad j = 1, ..., b_i \quad (15)$$

where μ is a constant parameter, A_i and E_{ij} are independent normal random variables with zero means and variances σ_A^2 and σ_E^2 respectively.

In this section we will present three new methods for obtaining confidence intervals for σ_A^2 for the model in (15). The three methods are

Method A: A modification of TW procedure.
Method B:A modification of method 4.
Method C: A modification of method 5.

Any of the five methods presented in section 2 can be modified for the unbalanced case but we chose TW's method rather than Howe's or Bulmer-Moriguti's method to modify because it has been shown that of the three methods, although Howe's method is the best, TW's method is "almost" as good as Howe's method and in many cases is "as" good. Also it has been proved by Wang (1990) that the confidence coefficient using the TW method is $\geq 1-\alpha$. In addition the TW formula is the simplest of the three methods to compute . We also examine methods B and C for the unbalanced case since they have not been previously examined.

First we state three theorms that will be used to derive methods A, B, and C.

Theorem 1

In the unbalanced model $Y_{ij} = \mu + A_i + E_{ij}$ let $\mathbf{Y} = [\bar{Y}_1, \bar{Y}_2, ..., \bar{Y}_a]^T$ where $\bar{Y}_i = (1/b_i)\sum_{j=1}^{b_i} Y_{ij}$. Then $\mathbf{Y} \sim MVN(\mu\mathbf{1}, \mathbf{\Sigma})$ where $\mathbf{\Sigma} = \sigma_A^2\mathbf{I} + \sigma_E^2\mathbf{K}$ and where $\mathbf{K}$ is a diagonal matrix with $1/b_i$ on the ith diagonal.

Theorem 2

In the unbalanced model let $\mathbf{Y'AY} = \mathbf{Y'}[\mathbf{I} - (1/a)\mathbf{J}]\mathbf{Y} = \sum_{i=1}^{a}(\bar{Y}_i - \bar{Y})^2$ where $\bar{Y} = (1/a)\sum_{i=1}^{a}\bar{Y}_i$; then $\mathbf{Y'AY}$ is distributed as $\sum_{i=1}^{a-1}\gamma_i V_i$ where γ_i are the non-zero characteristic roots of $\mathbf{A\Sigma}$ and V_i are independent chi-squared random variables where each has one degree of freedom. For a discussion of this theorem see Graybill (1976).

Theorem 3

If γ_{min} and γ_{max} are the minimum and maximum non-zero characteristic roots of $\mathbf{A\Sigma}$, then

$$\theta_{min} \leq \gamma_{min} \leq \gamma_{max} \leq \theta_{max}, \tag{16}$$

where θ_{min} and θ_{max} are the minimum and maximum characteristic roots of $\mathbf{\Sigma}$.

We will outline a proof of this theorem.

$\mathbf{A\Sigma} = \sigma_A^2\mathbf{A} + \sigma_E^2\mathbf{AK}$ where $\mathbf{A} = \mathbf{I} - (1/a)\mathbf{J}$ is an idempotent matrix of rank $a-1$. Thus there exists an orthogonal matrix $\mathbf{Q}$ such that

$$\mathbf{Q'AQ} = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{a-1} \end{bmatrix}$$

Let $\Upsilon(\mathbf{A\Sigma})$ be the characteristic roots of $\mathbf{A\Sigma}$, then we have the following. $\Upsilon\mathbf{A\Sigma}) = \Upsilon(\mathbf{A\Sigma A}) = \Upsilon(\mathbf{Q'AQQ'\Sigma QQ'AQ}) = \Upsilon(\mathbf{G})$ where $\mathbf{G}$ is given by

$$\mathbf{G} = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma_2} \end{bmatrix}$$

where $\mathbf{\Sigma_2}$ is a principal $(a-1)\times(a-1)$ submatrix of $\mathbf{Q'\Sigma Q}$.
By the seperation theorem (Wilkinson 1972) if $\theta_1 \leq \theta_2 \leq ... \leq \theta_a$ are the characteristic roots of $\mathbf{\Sigma}$ and $\gamma_2 \leq ... \leq \gamma_a$ are the characteristic roots of $\mathbf{\Sigma_2}$ then

$$\theta_1 \leq \gamma_2 \leq \theta_2 ... \leq \gamma_a \leq \theta_a.$$

Since the non-zero characteristic roots of $\mathbf{A\Sigma}$ are the same as the characteristic roots of $\mathbf{\Sigma_2}$ it follows that

$$\theta_{min} \leq \gamma_{min} \leq \gamma_{max} \leq \theta_{max}. \tag{17}$$

This completes the proof.

Let $r_1 = a-1$ and we have
$1-\alpha = P[\sum_{i=1}^{r_1} V_i \leq r_1 F_{1-\alpha:r_1\infty}] = P[\sum_{i=1}^{r_1} \gamma_{max} V_i \leq \gamma_{max} r_1 F_{1-\alpha:r_1,\infty}]$.
But $\sum \gamma_i V_i \leq \sum \gamma_{max} V_i \leq \sum \theta_{max} V_i$ so we get $P[\sum_{i=1}^{r_1} \gamma_i V_i \leq \theta_{max} r_1 F_{1-\alpha,r_1,\infty}] \geq 1-\alpha$.
Substituting $\mathbf{Y'AY}$ for $\sum \gamma_i V_i$ we get

$$P(\mathbf{Y'AY}/\theta_{max} \leq r_1 F_{1-\alpha,r_1,\infty}) \geq 1-\alpha. \tag{18}$$

Similarly
$1-\alpha = P[\sum_{i=1}^{r_1} V_i \geq r_1 F_{1-\alpha:r_1,\infty}] = P[\sum_{i=1}^{r_1} \gamma_{min} V_i \geq \gamma_{min} r_1 F_{1-\alpha:r_1,\infty}]$.
But $\sum \gamma_i V_i \geq \sum \gamma_{min} V_i \geq \sum \theta_{min} V_i$, so we get $P[\sum_{i=1}^{r_1} \gamma_i V_i \geq \theta_{min} r_1 F_{1-\alpha,r_1,\infty}] \geq 1-\alpha$.
Substituting $\mathbf{Y'AY}$ for $\sum \gamma_i V_i$ we get

$$P(\mathbf{Y'AY}/\theta_{min} \geq r_1 F_{1-\alpha,r_1,\infty}) \geq 1-\alpha. \qquad (19)$$

But $\mathbf{\Sigma} = \sigma_A^2 \mathbf{I} + \sigma_E^2 \mathbf{K}$ where $\mathbf{K} = \text{diag}(1/b_1, 1/b_2, ..., 1/b_a)$. Hence the charcteristic roots of $\mathbf{\Sigma}$ are

$$\sigma_A^2 + \sigma_E^2/b_i \qquad \text{for } i = 1, ..., a$$

Thus

$$\theta_{min} = \sigma_A^2 + \sigma_E^2/M \qquad (20)$$

and

$$\theta_{max} = \sigma_A^2 + \sigma_E^2/m \qquad (21)$$

where m and M are the minimum and maximum of $b_i s$ respectively for $i = 1, ..., a$. Hence

$$P(\mathbf{Y'AY}/r_1 F_{1-\alpha,r_1,\infty} \leq \sigma_A^2 + \sigma_E^2/m) \geq 1-\alpha \qquad (22)$$

and

$$P(\mathbf{Y'AY}/r_1 F_{\alpha,r_1,\infty} \geq \sigma_A^2 + \sigma_E^2/M) \geq 1-\alpha. \qquad (23)$$

We use (22) and (23) to derive the three methods A, B and C for obtaining confidence intervals for σ_A^2.

Method A - Modification of TW procedure

Replacing S_A^2 with $\mathbf{Y'AY}$ in equations (2), (3) and using the minimum of $E(\mathbf{Y'AY})$ instead of $E(S_A^2)$ we get a modified version of the lower bound of the TW formula. Using the maximum of $E(\mathbf{Y'AY})$ instead of $E(S_A^2)$ we get a modified form of the upper bound of the TW formula. The lower and upper bounds are given by L_A and U_A where

$$L_A = [\mathbf{Y'AY}/r_1 - (F_{1-\alpha:r_1,r_2} S_E^2)/M]/F_{1-\alpha:r_1,\infty} \qquad (24)$$

and

$$U_A = [\mathbf{Y'AY}/r_1 - (F_{\alpha:r_1,r_2} S_E^2)/m]/F_{\alpha:r_1,\infty} \qquad (25)$$

where $S_E^2 = \sum_{i=1}^{a} \sum_{j=1}^{b_j} (Y_{ij} - \bar{Y}_i)^2 / r_2$, $r_1 = a - 1$, $r_2 = b. - a$, $b. = \sum_i b_i$ and $\mathbf{Y'AY} = \sum_i (\bar{Y}_i - \bar{Y})^2$.

Method B - Modification of Method 4

In order to modify method 4 we will take the intersection of (22) and (23) with equation (10), the points of intersection will give us the lower and upper bounds for σ_A^2 as L_B and U_B respectively where.

$$L_B = \mathbf{Y'AY}/r_1 F_{1-\alpha:r_1,\infty} - S_E^2 / M F_{\alpha:r_2,\infty} \tag{26}$$

$$U_B = \mathbf{Y'AY}/r_1 F_{\alpha:r_1,\infty} - S_E^2 / m F_{1-\alpha:r_2,\infty} \tag{27}$$

where $r_2 = b. - a$.

Method C - Modification of Method 5

In the unbalanced one-way model Wald (1940) gave a procedure for finding exact lower and upper confidence bounds for $\tau = \sigma_A^2/\sigma_E^2$. This method requires the solution of two nonlinear equations. The $1 - \alpha$ lower and upper confidence bounds given by Wald are denoted by L_W and U_W respectively where

$$P[L_W \le \tau \le U_W] = 1 - \alpha \tag{28}$$

where L_W is the root of the equation

$$f(\tau) = F_{\alpha:r_1,r_2} \tag{29}$$

and U_W is the root of the equation

$$f(\tau) = F_{1-\alpha:r_1,r_2} \tag{30}$$

where

$$f(\tau) = \sum_{i=1}^{a} w_i (\bar{Y}_i - \sum_i w_i \bar{Y}_i / \sum_i w_i)^2 / (a-1) S_E^2 \tag{31}$$

and where $w_i = b_i / (1 + b_i \tau)$.
By the Bonferroni method the intersection of (28) with equation (10) gives the lower and upper bounds for σ_A^2 as L_C and U_C where

$$L_C = (L_W)(L_3) \tag{32}$$

$$U_C = (U_W)(U_3) \tag{33}$$

4 Evaluation of the Procedures

The only known method for obtaining confidence intervals with confidence coefficients $\geq 1-\alpha$ for the unbalanced model in (15) is to discard data at random in each cell so that all cells contain m=min(b_i) observations and use the TW method for the resulting balanced model. We will denote this method as the discarded TW method, DTW.

Simulation was used to evaluate methods A, B, and C, by computing confidence coefficients and expected widths. These were compared with the DTW procedure as the standard. It can be shown that the confidence coefficients for methods A, B, C and DTW depend on the unknown parameters σ_A^2 and σ_E^2 only through ρ where $\rho = \sigma_A^2/(\sigma_A^2+\sigma_E^2)$. Thus the confidence coefficients depend on b_i, a, $1-\alpha$, which are known and ρ which is unknown. For details see Fayyad (1993).

Simulations were used to evaluate and compare the methods. The values of ρ were taken to be 0.01(0.01)0.1, 0.1(0.1)0.9, 0.99. The values of a used were 3,4,8 and 10; various values of b_i were used for each value of a; $1-\alpha$ was taken as 0.90, 0.95, 0.99. Tables (1), (2), (3) and (4) show results for $1-\alpha = 0.95$. For details of simulation and results for $1-\alpha = 0.90$ and $1-\alpha = 0.99$ you can consult Fayyad (1993).

The 'expected widths' $E|L-\sigma_A^2|/(\sigma_A^2+\sigma_E^2)$ were used for the lower bounds, and $E|U-\sigma_A^2|/(\sigma_A^2+\sigma_E^2)$ for the upper bounds. The average widths were computed for methods A, B, C and DTW. The ratio of the average width for methods A, B, and C to the average width using the DTW procedure was computed. Thus to evaluate procedures A, B and C for lower bounds we computed

$$E|L_A-\sigma_A^2|/E|L_{DTW}-\sigma_A^2|,$$
$$E|L_B-\sigma_A^2|/E|L_{DTW}-\sigma_A^2|$$

and

$$E|L_C-\sigma_A^2|/E|L_{DTW}-\sigma_A^2|.$$

The same was done for upper bounds.

Tables (1), (2), (3) and (4) summarize the results obtained. Tables (1) and (2) show the ranges of confidence coefficients where the confidence coefficients are calculated for each value of ρ and the minimum and maximum confidence coefficients are given. Tables (3) and (4) give the minimum and maximum values of ratios of expected widths where the ratios are calculated for each value of ρ. From Tables (1) and (2) the confidence coefficients for upper and lower confidence bounds are $\geq 1-\alpha$ except for one case where method C does not attain the stated confidence coefficient for the upper bound. From Table (3) for moderately unbalanced data method A gives the lowest expected width.

Once the data becomes very unbalanced, method A gives larger expected widths than the DTW method; however method C has lower expected widths than method A in some of the cases but it still has slightly larger expected width than the DTW procedure. These unbalanced cases are extreme and would very rarely occur in practical situations; hence for practical situations method A seems to be the 'best' for upper bounds. For the lower bounds (Table 4) method A has the smallest expected widths for balanced, moderately unbalanced and very unbalanced designs, hence method A seems to be the best of the four procedures for lower bounds. So overall we recommend that method A be used to compute upper, lower and two sided confidence intervals for σ_A^2 in the unbalanced one-way variance components model.

5 An Example

Swallow and Searle (1978) presented the data shown in the Table below in which five groups of vegetable oil were randomly selected from a moving production line and weighted. We will compute lower, and upper confidence bounds for σ_A^2, the variance of a single weighing, using method A and method B. The data are used to calculate $\mathbf{Y'AY} = 0.01425$ and $S_E^2 = 0.00214$, and these were substituted into formulas (24), (25), (26) and (27). The values of L_A, L_B, U_A and U_B respectively were obtained for $1 - \alpha = 0.95$. The values are: $L_A = 0.00089$, $U_A = 0.019$, $L_B = 0.00047$ and $U_B = 0.019$.

Weights of Bottles (in ounces)

	Group				
	1	2	3	4	5
	15.70	15.69	15.75	15.68	15.65
	15.68	15.72	15.82	15.66	15.60
	15.64		15.75	15.59	
	15.60		15.71		
			15.84		
$\bar{Y}_i$	15.655	15.7	15.774	15.643	15.625

Table 1

Ranges Of Confidence Coefficients For Upper Bounds as ρ varies from 0 to 1.

$1 - \alpha = 0.95$

a	b_i	Range for Method C	Range for Method B	Range for DTW	Range for Method A
3	2 3 4	0.951-0.985	0.951-0.982	0.949-0.952	0.950-0.955
3	2 10 20	0.952-0.970	0.951-0.988	0.951-0.953	0.951-0.961
3	2 2 1000	0.948-0.955	0.951-0.999	0.951-0.951	0.950-0.971
3	10 10 10	0.953-0.971	0.951-0.966	0.951-0.953	0.951-0.952
3	10 20 30	0.952-0.965	0.951-0.973	0.951-0.952	0.951-0.956
4	2 2 2 2	0.952-0.996	0.954-0.983	0.951-0.953	0.951-0.955
4	2 2 2 3	0.952-0.994	0.953-0.987	0.951-0.953	0.952-0.957
4	2 2 4 4	0.951-0.990	0.954-0.988	0.952-0.954	0.951-0.958
4	2 2 100 100	0.952-0.962	0.954-0.998	0.953-0.953	0.954-0.976
4	10 10 10 11	0.956-0.973	0.953-0.971	0.952-0.953	0.953-0.954
4	10 10 10 100	0.952-0.964	0.953-0.994	0.953-0.953	0.953-0.971
8	2 3 4 5 6 7 8 9	0.950-0.983	0.954-0.994	0.946-0.957	0.954-0.964
8	2 2 2 2 2 2 2 1000	0.938-0.960	0.954-1.000	0.953-0.954	0.946-0.988
8	10 10 10 10 10 10 10 10	0.956-0.975	0.954-0.973	0.952-0.954	0.954-0.956
8	10 20 30 40 50 60 70 80	0.955-0.963	0.954-0.987	0.949-0.954	0.954-0.968
8	50 50 50 50 50 50 50 100	0.957-0.963	0.954-0.980	0.953-0.954	0.954-0.966
8	50 50 50 50 50 50 100 100	0.957-0.963	0.954-0.973	0.952-0.954	0.954-0.960
10	2 3 4 5 6 7 8 9 10 11	0.954-0.979	0.945-0.997	0.944-0.948	0.945-0.969
10	2 2 2 2 2 2 2 2 2 1000	0.944-0.958	0.945-1.00	0.945-0.947	0.945-0.993
10	10 10 10 10 10 10 10 10 10 10	0.951-0.977	0.945-0.967	0.945-0.950	0.945-0.950
10	10 20 30 40 50 60 70 80 90 100	0.954-0.960	0.945-0.995	0.944-0.946	0.945-0.974
10	20 20 20 20 20 20 20 20 20 1000	0.947-0.957	0.945-1.00	0.945-0.946	0.945-0.991
10	50 50 50 50 50 50 50 50 50 60	0.952-0.960	0.945-0.967	0.943-0.945	0.945-0.952
10	50 50 50 50 50 50 50 50 50 100	0.953-0.960	0.945-0.982	0.945-0.947	0.945-0.963

Table 2

Ranges Of Confidence Coefficients For Lower Bounds as ρ varies from 0 to 1.

$1 - \alpha = 0.95$

a	b_i	Range for Method C	Range for Method B	Range for DTW	Range for Method A
3	2 3 4	0.963-1.000	0.952-0.996	0.951-0.970	0.951-0.973
3	2 10 20	0.960-0.992	0.951-0.999	0.950-0.953	0.951-0.978
3	2 2 1000	0.951-0.961	0.951-0.986	0.950-0.952	0.951-0.966
3	10 10 10	0.959-0.992	0.951-0.984	0.951-0.956	0.950-0.955
3	10 20 30	0.960-0.984	0.951-0.997	0.951-0.952	0.951-0.970
4	2 2 2 2	0.963-1.000	0.951-0.993	0.947-0.971	0.949-0.974
4	2 2 2 3	0.963-1.000	0.951-0.993	0.947-0.971	0.950-0.971
4	2 2 4 4	0.961-1.000	0.950-0.996	0.949-0.965	0.949-0.973
4	2 2 100 100	0.953-0.971	0.949-0.995	0.948-0.950	0.949-0.976
4	10 10 10 11	0.956-0.992	0.948-0.984	0.947-0.952	0.947-0.951
4	10 10 10 100	0.954-0.977	0.948-0.987	0.947-0.949	0.948-0.959
8	2 3 4 5 6 7 8 9	0.957-0.998	0.952-1.000	0.949-0.956	0.951-0.987
8	2 2 2 2 2 2 2 1000	0.951-0.966	0.950-0.978	0.948-0.950	0.950-0.959
8	10 10 10 10 10 10 10 10	0.961-0.992	0.950-0.984	0.950-0.953	0.950-0.954
8	50 50 50 50 50 50 50 100	0.960-0.975	0.950-0.973	0.949-0.950	0.950-0.956
8	50 50 50 50 100 100 100 100	0.961-0.971	0.950-0.985	0.949-0.950	0.950-0.965
8	10 20 30 40 50 60 70 80	0.959-0.976	0.950-1.000	0.949-0.950	0.950-0.992
10	2 3 4 5 6 7 8 9 10 11	0.955-0.996	0.951-1.000	0.946-0.955	0.950-0.991
10	2 2 2 2 2 2 2 2 2 1000	0.950-0.968	0.950-0.979	0.949-0.950	0.950-0.957
10	10 10 10 10 10 10 10 10 10 10	0.958-0.992	0.950-0.982	0.950-0.953	0.950-0.953
10	10 20 30 40 50 60 70 80 90 100	0.959-0.973	0.950-1.000	0.948-0.951	0.950-0.994
10	20 20 20 20 20 20 20 20 20 1000	0.953-0.966	0.949-0.976	0.949-0.951	0.951-0.956
10	50 50 50 50 50 50 50 50 50 60	0.959-0.974	0.950-0.967	0.950-0.951	0.950-0.952
10	50 50 50 50 50 50 50 50 50 100	0.960-0.973	0.950-0.972	0.950-0.951	0.950-0.955

Table 3

Ranges Of Ratios Of Average Widths Of Each Of The Three Methods To The Expected Width Using DTW Method For Upper Bounds for $1-\alpha=0.95$

a	b_i	Range for Ratio Method C to DTW	Range for Ratio Method B to DTW	Range for Ratio Method A to DTW
3	2 3 4	2.652-3.766	0.754-0.999	0.739-0.999
3	2 10 20	0.730-1.662	0.469-0.997	0.466-0.997
3	2 2 1000	0.728-1.080	0.713-0.999	0.713-0.999
3	10 10 10	1.670-1.702	1.000-1.015	0.999-1.000
3	10 20 30	0.911-1.420	0.667-1.000	0.662-1.000
4	2 2 2 2	6.003-6.642	1.000-1.058	0.996-1.000
4	2 2 2 3	4.212-5.061	0.981-1.000	0.934-1.000
4	2 2 4 4	2.276-3.284	0.808-0.999	0.775-0.999
4	2 2 100 100	0.590-1.210	0.580-0.998	0.577-0.998
4	10 10 10 11	1.520-1.617	1.000-1.014	0.983-1.000
4	10 10 10 100	0.981-1.280	0.870-1.000	0.863-1.000
8	2 3 4 5 6 7 8 9	0.717-1.849	0.578-0.998	0.528-0.964
8	2 2 2 2 2 2 2 1000	0.913-1.110	1.001-1.244	1.001-1.228
8	10 10 10 10 10 10 10 10	1.385-1.524	1.000-1.078	0.999-1.000
8	10 20 30 40 50 60 70 80	0.432-1.198	0.482-0.999	0.462-0.999
8	50 50 50 50 50 50 50 100	1.092-1.174	1.000-1.092	1.000-1.061
8	50 50 50 50 100 100 100 100	0.927-1.148	0.910-1.000	0.888-1.000
10	2 3 4 5 6 7 8 9 10 11	0.564-1.693	0.575-0.998	0.508-0.997
10	2 2 2 2 2 2 2 2 2 1000	0.939-1.119	1.002-1.400	1.002-1.379
10	10 10 10 10 10 10 10 10 10 10	1.344-1.500	1.000-1.092	1.000-1.003
10	10 20 30 40 50 60 70 80 90 100	0.369-1.169	0.491-0.999	0.463-0.999
10	20 20 20 20 20 20 20 20 20 1000	0.967-1.111	1.000-1.324	1.000-1.312
10	50 50 50 50 50 50 50 50 50 60	1.125-1.178	1.000-1.068	1.000-1.028
10	50 50 50 50 50 50 50 50 50 100	1.091-1.170	1.000-1.141	1.000-1.100

Table 4

Ranges Of Ratios Of Average Widths Of Each Of The Three Methods To The Expected Width Using DTW Method For Lower Bounds for $1 - \alpha = 0.95$

a	b_i	Range for Ratio Method C to DTW	Range for Ratio Method B to DTW	Range for Ratio Method A to DTW
3	2 3 4	0.616-1.280	0.600-1.066	0.808-0.996
3	2 10 20	0.623-1.135	0.561-1.086	0.637-1.000
3	2 2 1000	0.797-1.020	0.673-1.033	0.766-1.000
3	10 10 10	0.932-1.144	0.931-1.058	1.000-1.010
3	10 20 30	0.899-1.097	0.895-1.065	0.907-1.000
	2 2 2 2	0.612-1.433	0.696-1.110	0.996-1.003
4	2 2 2 3	0.627-1.402	0.651-1.091	0.932-0.999
4	2 2 4 4	0.651-1.342	0.622-1.096	0.819-0.998
4	2 2 100 100	0.666-1.065	0.615-1.078	0.671-1.002
4	10 10 10 11	0.934-1.168	0.932-1.068	0.992-1.000
4	10 10 10 100	0.928-1.087	0.927-1.059	0.943-1.001
8	2 3 4 5 6 7 8 9	0.722-1.319	0.679-1.189	0.718-1.008
8	2 2 2 2 2 2 2 1000	0.892-1.055	0.807-1.055	0.899-1.003
8	10 10 10 10 10 10 10 10	0.966-1.227	0.971-1.092	0.998-1.000
8	50 50 50 50 50 50 50 100	0.990-1.086	1.000-1.055	0.991-1.001
8	50 50 50 50 100 100 100 100	0.955-1.074	1.000-1.093	0.975-1.007
8	10 20 30 40 50 60 70 80	0.824-1.097	0.961-1.176	0.916-1.020
10	2 3 4 5 6 7 8 9 10 11	0.705-1.304	0.677-1.231	0.687-1.016
10	2 2 2 2 2 2 2 2 2 1000	0.897-1.065	0.822-1.063	0.907-1.003
10	10 10 10 10 10 10 10 10 10 10	0.973-1.239	0.977-1.098	0.999-1.003
10	10 20 30 40 50 60 70 80 90 100	0.770-1.091	0.968-1.215	0.916-1.034
10	20 20 20 20 20 20 20 20 20 1000	0.953-1.061	1.000-1.061	0.967-1.003
10	50 50 50 50 50 50 50 50 50 60	1.000-1.096	1.000-1.054	0.999-1.001
10	50 50 50 50 50 50 50 50 50 100	0.993-1.092	1.000-1.061	0.992-1.002

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