Underlying Spaces of Noncommutative Schemes

Introduction

The general philosophy adopted in noncommutative algebraic geometry is based on identifying 'spaces' with categories regarded as categories of quasi-coherent or coherent sheaves, or derived categories of those. The purpose is to describe geometric notions in terms of related categories and functors between them and then use these descriptions for defining and studying new, noncommutative, 'spaces'. It is similar to noncommutative differential geometry started by Alain Connes, where manifolds are replaced by algebras.

In this approach only a half of the picture survives: geometric spaces become 'virtual'.

Most of commutative algebraic geometry, however, is based upon studying geometric spaces (see [Gr1], [Gr2], [Gr5], [Gr6], [Gr7], [Gr8], [GrD]). Grothendieck's local study of schemes turns commutative algebra into a part of geometry. One of the goals of noncommutative algebraic geometry is to do the same for noncommutative algebra, in particular, to develop geometric tools and intuition for studying representations of associative algebras.

This work is a step in that direction. One of its purposes is to define underlying topological spaces of noncommutative schemes and some other noncommutative 'spaces' of interest. A natural framework for this is provided by *quasi-topological 'spaces'* represented by abelian categories (defined in 1.5). The main building block used here is the *spectrum* introduced in [R1] (see also [R, Ch.3]). This spectrum is defined for an arbitrary abelian category and is directly related with representation theory: isomorphism classes of simple objects are closed points of the spectrum. The spectrum of the category of modules over an arbitrary commutative ring is naturally isomorphic to the prime spectrum of this ring.

Below follows a brief outline of the content.

First three sections contain preliminaries on 'spaces' and basic spectra used in the sequel. Section 1 starts with a short dictionary of categoric (virtual) geometry. 'Spaces' considered here are represented by abelian categories; and morphisms of 'spaces', $X \longrightarrow Y$, are represented by additive (inverse image) functors, $C_Y \longrightarrow C_X$, between the corresponding categories. We recall the notion of a quasi-pretopology and define the category of quasi-topological 'spaces'. Its objects are pairs (X, τ_X), where X is a 'space' and τ_X is a set of *covers*. Covers considered in this work are conservative families of exact localizations (i.e. the set of their inverse image functors is a conservative family of exact localizations).

In Section 2, we define the spectrum of a 'space' X as the spectrum (in the sense of [R1], or [R, Ch.3]) of the abelian category representing X and sketch some of its properties.

We remind the notion of a Serre subcategory, as it is introduced in [R1], and define Serre localizations as exact localizations whose kernels are Serre subcategories.

Section 3 contains preliminaries on *local* 'spaces', *complete spectrum*, and the *S-spectrum* (the main references are [R, Ch.3], and [R, Ch.6], where the S-spectrum is called *flat spectrum*). A space X is *local* if the spectrum of X has a unique closed point which

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belongs to the support of every nonzero object of the category C_X. If C_X is the category of modules over a commutative ring R, then X is local iff the ring R is local.

The *complete spectrum* of X is a preordered set, $\mathbf{Spec}^1(X)$, whose elements are thick subcategories $\mathcal{P}$ such that the quotient category $C_X/\mathcal{P}$ is local. The S-spectrum of a 'space' X is a preordered set, $\mathbf{Spec}^-(X)$, formed by all Serre subcategories of C_X which belong to the complete spectrum. The spectrum $\mathbf{Spec}(X)$ is naturally embedded into $\mathbf{Spec}^-(X)$, and for every exact localization $U \longrightarrow X$ (an 'open immersion' in an appropriate sense), the spectrum $\mathbf{Spec}(U)$ is naturally embedded into $\mathbf{Spec}^1(X)$. Note by passing that this embedding is used to formulate the *local property* of the spectrum: for any conservative set, $\{U_i \to X \mid i \in J\}$, of exact localizations, $\mathbf{Spec}(X) \subseteq \bigcup_{i\in J} \mathbf{Spec}(U_i)$.

All spectra which appear in this work contain $\mathbf{Spec}(X)$ and are contained in $\mathbf{Spec}^1(X)$.

It is argued in [R, Ch.6] that if the category C_X is locally noetherian, then $\mathbf{Spec}^-(X)$ is naturally isomorphic to the *Gabriel spectrum* of the category C_X [Gab]. Recall that points of the Gabriel spectrum are isomorphism classes of indecomposable injectives. Even in the case of a locally noetherian 'space', the (image of the) spectrum $\mathbf{Spec}(X)$ is, usually, a small part of $\mathbf{Spec}^-(X)$.

In Section 4 we introduce the quasi-pretopology of Serre localizations and show that its finite covers form a Grothendieck pretopology on the subcategory whose objects are arbitrary 'spaces' X (such that C_X is an abelian category) and arrows are morphisms of 'spaces' whose inverse image functors are Serre localizations. Using this fact, we establish the *local property* of the S-spectrum: if a cover $\{U_i \to X \mid i \in J\}$ by Serre localizations contains a finite subcover, then $\mathbf{Spec}^-(X) = \bigcup_{i\in J} \mathbf{Spec}^-(U_i)$. We deduce from these assertions similar facts about complete spectrum and exact localizations: finite covers by exact localizations form a Grothendieck pretopology, and if a cover $\{U_i \to X \mid i \in J\}$ by exact localizations contains a finite subcover, then $\mathbf{Spec}^1(X) = \bigcup_{i\in J} \mathbf{Spec}^1(U_i)$.

In Section 5, we associate with every quasi-topological 'space' (X, τ_X) a topological space, $\mathbf{Spec}(X, \tau_X)$ which we call the *upper spectrum* of (X, τ_X). Points of $\mathbf{Spec}(X, \tau_X)$ are points $\mathbf{P}$ of the complete spectrum of X such that $\mathbf{P}$ belongs to $\mathbf{Spec}(U)$ for some element of a τ_X-cover $U \longrightarrow X$. If the quasi-pretopology τ_X is trivial (i.e. it consists only of identical morphism), then $\mathbf{Spec}(X, \tau_X)$ coincides with $\mathbf{Spec}(X)$. If τ_X is the quasi-pretopology of Serre localizations mentioned above, or even its finite version, then $\mathbf{Spec}(X, \tau_X)$ is the S-spectrum, $\mathbf{Spec}^-(X)$, of the 'space' X. If τ is (the finite version of) the quasi-pretopology of exact localizations, then $\mathbf{Spec}(X, \tau_X)$ coincides with the complete spectrum of X. If a quasi-pretopology $\widetilde{\tau}_X$ is a refinement of τ_X, then $\mathbf{Spec}(X, \tau_X) \subseteq \mathbf{Spec}(X, \widetilde{\tau}_X)$. In particular, $\mathbf{Spec}(X) \subseteq \mathbf{Spec}(X, \tau_X) \subseteq \mathbf{Spec}^1(X)$.

We show that, under a certain natural condition on the quasi-pretopology τ_X, the equality $\mathbf{Spec}(X, \tau_X) = \bigcup_{i\in J} \mathbf{Spec}(U_i, \tau_{U_i})$ holds for any τ-cover $\{U_i \to X \mid i \in J\}$. Here τ_{U_i} is the quasi-pretopology on U_i induced by the quasi-pretopology τ_X.

In Section 6, we introduce the *lower spectrum*, otherwise called the *combinatorial spectrum*, of a quasi-topological 'space' (X, τ_X). For a cover $\mathcal{U} = \{U_i \to X \mid i \in J\}$, we define the *lower spectrum of* $\mathcal{U}$ as a subset, $\mathbf{Spec}_{\wp}\mathcal{U}$, of all points $\mathbf{P}$ of the complete spectrum of X such that, whenever $\mathbf{P}$ belongs to the complete spectrum of some element, U_i, of the cover, it belongs to its spectrum, $\mathbf{Spec}(U_i)$.

The combinatorial spectrum, $\mathbf{Spec}_{\wp}(X,\tau_X)$ of a quasi-topological 'space' (X,τ_X) is a subspace of the complete spectrum of X formed by the points which *locally* belong to the spectrum. In precise terms, $\mathbf{Spec}_{\wp}(X,\tau_X)$ is the colimit of $\mathbf{Spec}_{\wp}\mathcal{U}$, where $\mathcal{U}$ runs through τ_X-covers of X and morphisms are induced by refinements. We have the inclusions of spectra

$$\mathbf{Spec}(X) \hookrightarrow \mathbf{Spec}_{\wp}(X,\tau_X) \hookrightarrow \mathbf{Spec}(X,\tau_X) \hookrightarrow \mathbf{Spec}^1(X). \qquad (\star)$$

Let $\mathbf{Spec}^-_{\wp}\mathcal{U}$ denote the intersection of $\mathbf{Spec}_{\wp}\mathcal{U}$ with the S-spectrum of X. We show that for any finite cover $\mathcal{U}$ of X, the natural embedding $\mathbf{Spec}(X) \hookrightarrow \mathbf{Spec}^-_{\wp}\mathcal{U}$ is an isomorphism. This implies that the canonical map $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^-_{\wp}(X,\tau_X)$ is an isomorphism, if (X,τ_X) is quasi-compact, i.e. if any τ_X-cover of X has a finite subcover.

More generally, if X has a τ_X-cover $\mathcal{U} = \{U_i \to X \mid i \in J\}$ such that all U_i are quasi-compact, then the canonical map $\mathbf{Spec}^-_{\wp}\mathcal{U} \longrightarrow \mathbf{Spec}^-_{\wp}(X,\tau_X)$ is an isomorphism.

If a cover $\mathcal{U}$ consists of Serre localizations (i.e. their kernels are Serre subcategories), then $\mathbf{Spec}^-_{\wp}\mathcal{U}$ coincides with $\mathbf{Spec}_{\wp}\mathcal{U}$. In particular, $\mathbf{Spec}^-_{\wp}(X,\tau_X) = \mathbf{Spec}_{\wp}(X,\tau_X)$ if τ_X has a base formed by Serre localizations.

In Section 7, we recall the notions of closed coimmersions and immersions (introduced in slightly different terms in [R], Ch.3) and use them to define some canonical quasi-pretopologies, in particular, the Zariski pretopology. If C_X is the category of left R-modules, then there is a bijective correspondence between isomorphism classes of closed immersions to X and two-sided ideals of the ring R. If C_X is the category of quasi-coherent sheaves on a scheme $\mathbf{X} = (\mathcal{X},\mathcal{O})$, then isomorphism classes of closed immersions to X are in one-to-one correspondence with closed subschemes of the scheme $\mathbf{X}$.

We apply the results of Sections 5, 6, and 7 to commutative schemes. If C_X is the category of quasi-coherent sheaves on a commutative scheme $\mathbf{X} = (\mathcal{X},\mathcal{O})$ and τ_X is the Zariski pretopology, then the upper spectrum, $\mathbf{Spec}(X,\tau_X)$, coincides with the combinatorial spectrum, $\mathbf{Spec}_{\wp}(X,\tau_X)$. If $\mathbf{X}$ is affine, then (by a result of Section 6) $\mathbf{Spec}_{\wp}(X,\tau_X) \simeq \mathbf{Spec}(X)$, and, by [R, Ch.3], $\mathbf{Spec}(X)$ is isomorphic to the prime spectrum of the ring $\Gamma\mathcal{O}$. It is easy to deduce from the latter fact that, in the case of an arbitrary scheme, $\mathbf{Spec}(X,\tau_X)$ (hence $\mathbf{Spec}_{\wp}(X,\tau_X)$) is isomorphic to the underlying topological space $\mathcal{X}$ of the scheme $\mathbf{X}$.

Suppose that a scheme $\mathbf{X} = (\mathcal{X},\mathcal{O})$ has a finite affine cover $\mathcal{U} = \{U_i \stackrel{u_i}{\to} \mathcal{X} \mid i \in J\}$ such that every immersion u_i has a direct image functor (for instance, $\mathbf{X}$ is quasi-compact and quasi-separated, or its underlying topological space, $\mathcal{X}$, is noetherian). Then it follows from results of Section 6 that the canonical embedding $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}_{\wp}(X,\tau_X)$ is an isomorphism. Therefore, $\mathbf{Spec}(X) \simeq \mathbf{Spec}_{\wp}(X,\tau_X) = \mathbf{Spec}(X,\tau_X) \simeq \mathcal{X}$.

This straightens a confusion created in [R2], where the isomorphism $\mathbf{Spec}(X) \simeq \mathcal{X}$ is claimed for an arbitrary scheme $(\mathcal{X},\mathcal{O})$. O. Gabber showed me (by producing an example) that the claim is not correct without some finiteness conditions (which are used, implicitly, in [R2]). An obvious remedy is to replace $\mathbf{Spec}(X)$ by the upper spectrum $\mathbf{Spec}(X,\tau^3_X)$, where τ^3_X is (a categorical incarnation of) the Zariski topology introduced in [R1]. This solution of the problem is not always satisfactory from the noncommutative point of view. The main reason is that the upper spectrum of $(X, \tau_X^{\mathfrak{Z}})$, where X is a noncommutative affine 'space', might be considerably larger than $\mathbf{Spec}(X)$.

The combinatorial spectrum seems to be a better, in a certain sense optimal, choice of the underlying topological space. If $C_X = R-mod$ and (X, τ_X) is quasi-compact, then the combinatorial spectrum of (X, τ_X) is isomorphic to $\mathbf{Spec}(X)$. By a result of [R1], Zariski pretopology on any affine noncommutative scheme is quasi-compact.

In Section 8, we associate with a quasi-topological 'space' (X, τ_X) and a choice of a spectrum, $\mathbf{Spec}^{\bullet}(X, \tau_X)$, realized as a subset of the complete spectrum and endowed with the topology induced by τ_X, a sheaf of commutative rings, $\mathcal{O}_X^{\mathfrak{c}}$, on $\mathbf{Spec}^{\bullet}(X, \tau_X)$. In particular, we associate with the 'space' (X, τ_X) the ringed topological spaces $(\mathbf{Spec}_{\wp}(X, \tau_X), \mathcal{O}_X^{\mathfrak{c}})$ and $(\mathbf{Spec}(X, \tau_X), \mathcal{O}_X^{\mathfrak{c}})$ called respectively the *lower* (or *combinatorial*) and the *upper geometric center* of (X, τ_X). Applying the facts of Section 6, we show that if C_X is the category of quasi-coherent sheaves on a scheme $\mathbf{X} = (\mathcal{X}, \mathcal{O})$, and τ_X is the categorical analog of Zariski topology discussed in Section 7, then the upper and lower geometric centra coincide and are isomorphic to the scheme $\mathbf{X}$.

In Section 9, we recall the notion of a noncommutative scheme and discuss shortly geometric realizations of noncommutative schemes based on the combinatorial spectrum and on the upper spectrum. Most of non-affine noncommutative 'spaces' studied in literature are analogues of **Cone** and **Proj** and some related examples of noncommutative schemes. We remind the notions of the **Cone** of a non-unital ring and the **Proj** of a $\mathcal{G}$-graded ring, where $\mathcal{G}$ is a monoid, and study the relations between their combinatorial spectra. This study involves the notion of $\mathcal{G}$*-spectrum*. One of the applications is the description of the combinatorial spectrum of the quantum flag variety of a semisimple Lie algebra in terms of the spectrum of its base affine space.

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1. Preliminaries on 'spaces'.

1.1. Categories and 'spaces'. As usual, Cat, or $Cat_{\mathfrak{U}}$, denotes the bicategory of categories which belong to a fixed universum $\mathfrak{U}$. We call objects of Cat^{op} 'spaces'. For any 'space' X, the corresponding category C_X is regarded as the category of quasi-coherent sheaves on X. We denote by f^* the functor $C_Y \longrightarrow C_X$ corresponding to a 1-morphism $X \xrightarrow{f} Y$ and call it the *inverse image functor* of f. For any $\mathfrak{U}$-category $\mathcal{A}$, we denote by $|\mathcal{A}|$ the corresponding object of Cat^{op} (the underlying 'space') defined by $C_{|\mathcal{A}|} = \mathcal{A}$.

We denote by $|Cat|^o$ the category having same objects as Cat^{op}. Morphisms from X to Y are isomorphism classes of functors $C_Y \longrightarrow C_X$. For a morphism $X \xrightarrow{f} Y$, we denote by f^* any functor $C_Y \longrightarrow C_X$ representing f and call it an *inverse image functor of the morphism* f. We shall write $f = [F]$ to indicate that f is a morphism having an inverse image functor F. The composition of morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ is defined by $g \circ f = [f^* \circ g^*]$. Thus, the map which assigns to each functor $C_Y \xrightarrow{F} C_X$ the morphism $X \xrightarrow{[F]} Y$ is a functor $Cat^{op} \longrightarrow |Cat|^o$ which turns Cat^{op} into a fibred category over $|Cat|^o$.

1.2. Localizations and conservative morphisms. Let Y be an object of $|Cat|^o$ and Σ a class of arrows of the category C_Y. We denote by $\Sigma^{-1}Y$ the object of $|Cat|^o$ such that the corresponding category coincides with (the standard realization of) the quotient of the category C_Y by Σ (cf. [GZ, 1.1]): $C_{\Sigma^{-1}Y} = \Sigma^{-1}C_Y$. The canonical *localization functor* $C_Y \xrightarrow{p_\Sigma^*} \Sigma^{-1}C_Y$ is regarded as an inverse image functor of a morphism, $\Sigma^{-1}Y \xrightarrow{p_\Sigma} Y$.

For any morphism $X \xrightarrow{f} Y$ in $|Cat|^o$, we denote by Σ_f the family of all arrows s of the category C_Y such that $f^*(s)$ is invertible (notice that Σ_f does not depend on the choice of an inverse image functor f^*). Thanks to the universal property of localizations, f^* is represented as the composition of the localization functor $p_f^* = p_{\Sigma_f}^* : C_Y \longrightarrow \Sigma_f^{-1}C_Y$ and a uniquely determined functor $\Sigma^{-1}C_Y \xrightarrow{f_{\mathfrak{c}}^*} C_X$. In other words, $f = p_f \circ f_{\mathfrak{c}}$ for a uniquely determined morphism $X \xrightarrow{f_{\mathfrak{c}}} \Sigma_f^{-1}Y$.

A morphism $X \xrightarrow{f} Y$ is called *conservative* if Σ_f consists of isomorphisms only, or, equivalently, p_f is an isomorphism.

A morphism $X \xrightarrow{f} Y$ is called a *localization* if $f_{\mathfrak{c}}$ is an isomorphism, i.e. the functor $f_{\mathfrak{c}}^*$ is an equivalence of categories.

Thus, $f = p_f \circ f_{\mathfrak{c}}$ is a decomposition of a morphism f into a localization and a conservative morphism.

1.3. Left exact, right exact, and exact morphisms. A morphism $X \xrightarrow{f} Y$ is called *right exact* (resp. *left exact*, resp. *exact*), if its inverse image functor preserves colimits (resp. limits, resp. both limits and colimits) of arbitrary finite diagrams.

The following assertion is a reformulation of Proposition 1.1.4 in [GZ].

1.3.1. Proposition. *Let $f = p_f \circ f_{\mathfrak{c}}$ be the canonical decomposition of a morphism $X \xrightarrow{f} Y$ into a conservative morphism $X \xrightarrow{f_{\mathfrak{c}}} \Sigma_f^{-1}Y$ and a localization $\Sigma_f^{-1}Y \xrightarrow{p_f} Y$.*

Suppose C_Y has finite limits (resp. finite colimits). Then f is left exact (resp. right exact) iff the class of arrows Σ_f is a left (resp. right) saturated multiplicative system. In this case both the localization p_f and the conservative morphism $f_{\mathfrak{c}}$ are left (resp. right) exact.

In particular, if the category C_Y has limits and colimits of finite diagrams, then f is exact iff both the localization p_f and the conservative component $f_{\mathfrak{c}}$ are exact. The exactness of p_f is equivalent to that Σ_f is a (left and right) multiplicative system.

1.4. Continuous morphisms and flat morphisms. A morphism f of $|Cat|^o$, or Cat^{op}, is called *continuous* if its inverse image functor has a right adjoint, f_*, which is called a *direct image functor* of f.

A morphism f is called *flat* if it is both exact and continuous.

One can show that a morphism f is continuous iff both the localization p_f and the conservative component $f_{\mathfrak{c}}$ are continuous.

1.5. Quasi-topological 'spaces'.

1.5.1. Quasi-pretopologies and quasi-presites. A *quasi-pretopology* on a category $\mathcal{A}$ is a function, τ, which assigns to each object X of $\mathcal{A}$ a set, τ_X, of covers; the latter are families of arrows $\{U_i \to X \mid i \in J\}$. We assume that any isomorphism is a cover and the composition of covers is a cover. The pair $(\mathcal{A}, \tau)$ is called a *quasi-presite*.

For any object X of $\mathcal{A}$, we denote by $\Sigma^\tau X$ the set of all arrows $U \longrightarrow X$ which belong to some cover of X. We denote by Σ^τ the union of all sets $\Sigma^\tau X,\ X \in Ob\mathcal{A}$. It follows that Σ^τ is closed under composition and contains all isomorphisms of the category $\mathcal{A}$, hence it defines a subcategory, $\mathcal{A}^\tau$, of $\mathcal{A}$.

Let $\Sigma_\tau X$ denote the class of all morphisms $Y \xrightarrow{f} X$ such that a pull-back $\mathcal{U} \times_X Y = \{U_i \times_X Y \longrightarrow Y \mid i \in J\}$ of any τ-cover $\mathcal{U} = \{U_i \to X \mid i \in J\}$ of X along f exists and is a τ-cover of Y. One can show that the class of arrows $\Sigma_\tau = \bigcup_{X \in Ob\mathcal{A}} \Sigma_\tau X$ contains all isomorphisms and is closed under the composition of morphisms, hence it defines a subcategory, $\mathcal{A}_\tau$, of the category $\mathcal{A}$.

For any $X \in Ob\mathcal{A}$, the pair (X, τ_X) is regarded as an analog of a topological space and will be called a *quasi-topological 'space'*. Morphisms from (Y, τ_Y) to (X, τ_X) are given by morphisms $Y \longrightarrow X$ which belong to Σ_τ. We denote thus defined category by $\mathfrak{Top}_{(\mathcal{A},\tau)}$.

A quasi-pretopology τ is a (Grothendieck) pretopology, if $\mathcal{A}_\tau = \mathcal{A}$. If τ is a pretopology on $\mathcal{A}$, the pair $(\mathcal{A}, \tau)$ is called a *presite*.

Notice that if $\Sigma^\tau \subseteq \Sigma_\tau$, then the pair $(\mathcal{A}_\tau, \tau|_{\mathcal{A}_\tau})$ is a presite.

1.5.2. The quasi-pretopology of exact localizations. Let $\mathcal{A}$ be the category $|Cat|^o$. We define the *quasi-pretopology of exact localizations*, $\tau^{\mathfrak{L}}$, as follows. For any $X \in Ob\mathcal{A}$, the set of covers $\tau^{\mathfrak{L}}_X$ consists of all conservative families $\{U_i \stackrel{u_i}{\to} X \mid i \in J\}$ of exact localizations; i.e. the set of inverse image functors $\{C_X \stackrel{u_i^*}{\to} C_{U_i} \mid i \in J\}$ is a conservative family of exact localizations.

1.6. 'Spaces' represented by abelian categories. We denote by $|\mathfrak{Ab}|^o$ the subcategory of the category $|Cat|^o$ whose objects are represented by abelian categories and morphisms have additive inverse image functors.

Note that if X is an object of $|\mathfrak{Ab}|^o$ and $U \xrightarrow{u} X$ is an exact localization, then the morphism u (in particular, the object U) belongs to the category $|\mathfrak{Ab}|^o$.

In fact, an inverse image functor, $C_X \stackrel{u^*}{\to} C_U$, of u being an exact localization means that the category C_U is equivalent to a quotient category of the abelian category C_X by a thick subcategory $Ker(u^*)$, hence C_U is an abelian category, and u^* is an additive functor.

1.6.1. Quasi-topological 'spaces'. In this work, we will consider quasi-presites $(\mathcal{A}, \tau)$ such that $\mathcal{A}$ is a subcategory of the category $|\mathfrak{Ab}|^o$ and for any $X \in Ob\mathcal{A}$, the set of covers τ_X is a subset of the set $\tau_X^{\mathfrak{L}}$ defined in 1.5.2; i.e. for every τ-cover $\{U_i \stackrel{u_i}{\to} X \mid i \in J\}$ is a conservative family of exact localizations.

Pairs (X, τ_X), $X \in Ob\mathcal{A}$, (objects of the category $\mathfrak{Top}_{(\mathcal{A},\tau)}$) are *quasi-topological 'spaces'* in the sense of 1.5.1. Any cover $\{U_i \stackrel{u_i}{\to} X \mid i \in J\}$ of a 'space' X can be represented by the set $\{Ker(u_i^*) \mid i \in J\}$ of kernels of inverse image functors. Exactness of the functor u_i^* means that its kernel is a thick subcategory of the category C_X. The fact that the family of functors $\{C_X \stackrel{u_i^*}{\to} C_{U_i} \mid i \in J\}$ is conservative means that the intersection of their kernels is the zero subcategory.

2. The spectrum.

2.1. The preorder $\succ$. Let X be an object of $|\mathfrak{Ab}|^o$, i.e. C_X is an abelian category. For any two objects, M and N, of C_X, we write $M \succ N$ if N is a subquotient of a finite coproduct of copies of M.

For any object M of the category C_X, we denote by $[M]$ (resp. $\langle M \rangle$) the full subcategory of C_X whose objects are all $L \in ObC_X$ such that $M \succ L$ (resp. $L \not\succ M$). It follows that $M \succ N \Leftrightarrow \langle N \rangle \subseteq \langle M \rangle \Leftrightarrow [N] \subseteq [M]$. In particular, M and N are equivalent with respect to $\succ$ (i.e. $M \succ N \succ M$) iff $\langle M \rangle = \langle N \rangle$, or $[M] = [N]$. Thus, the preorders $\big(\{[M] \mid M \in ObC_X\}, \supseteq\big)$ and $\big(\{\langle M \rangle \mid M \in ObC_X\}, \supseteq\big)$ are canonical realizations of the quotient of $(ObC_X, \succ)$ by the equivalence relation associated with $\succ$.

2.2. The spectrum Spec(X). We denote by $Spec(X)$ the family of all nonzero objects M of the category C_X such that $L \succ M$ for any nonzero subobject L of M, which is endowed with the preorder (induced by) $\succ$.

It follows from this definition that a nonzero object M of C_X belongs to $Spec(X)$ iff it is equivalent to any of its nonzero subobjects. In particular, every simple object of the category C_X belongs to $Spec(X)$.

We denote by $\mathbf{Spec}(X)$ the class of subcategories $\{[M] \mid M \in Spec(X)\}$ endowed with the preorder $\supseteq$. The preorder $\mathbf{Spec}(X)$ is called the *spectrum of* X. If C_X is equivalent to a small category, then $\mathbf{Spec}(X)$ is a preordered set.

Given elements $[M]$, $[P]$ of $\mathbf{Spec}(X)$, we say that $[P]$ is a *specialization* of $[M]$ if $[M] \supseteq [P]$, or, equivalently, $M \succ P$. We denote by $\tau^{\succ}$ the strongest topology on $\mathbf{Spec}(X)$ compatible with the preorder $\supseteq$. It can be described as follows: the closure of any subset W consists of specializations of all elements of W.

2.3. The spectrum and simple objects. Obviously, any simple object of the category C_X belongs to $Spec(X)$.

2.3.1. Proposition. *Let M, N be objects of the category C_X.*

(a) If M is simple and N is nonzero, the relation $M \succ N$ means that N is isomorphic to a coproduct of copies of M, in particular $N \succ M$, i.e. N and M are equivalent.

(b) If M and N are simple objects, then $M \succ N$ iff M and N are isomorphic.

Proof. Any nonzero subquotient of a semisimple object (a coproduct of simple objects) is semisimple too; hence the assertion (a). The assertion (b) is a consequence of (a). ∎

Let $\tau^{\succ}$ denote the topology on $\mathbf{Spec}(X)$ associated with the specialization preorder: the closure of $W \subseteq \mathbf{Spec}(X)$ consists of all $[M]$ such that $[M] \subseteq [M']$ for some $[M'] \in W$.

2.3.2. Proposition. *(a) The inclusion $Simple(X) \hookrightarrow Spec(X)$ induces an injection of the set of isomorphism classes of simple objects of C_X into the set of closed points of $(\mathbf{Spec}(X), \tau^{\succ})$.*

(b) If the category C_X has enough objects of finite type, then this injection is a bijection, i.e. every closed point of $(Spec(X), \tau^{\succ})$ is of the form $[M]$ for some simple object M of the category C_X.

Proof. (a) The assertion follows from 2.3.1.

(b) Having enough objects of finite type means that every object of C_X is the supremum of its subobjects of finite type. By a standard argument, this property implies that every nonzero object of C_X has a simple quotient. Suppose, $P \in Spec(X)$ is such that $[P]$ is a closed point. Let M be a simple quotient of P. Since $P \succ M$ and $[P]$ is closed, $M \succ P$, i.e. $[M] = [P]$. ∎

2.4. Topologizing and thick subcategories. A full subcategory, $\mathbb{T}$, of C_X is called *topologizing* if it is closed under finite coproducts and subquotients taken in C_X.

In particular, if M is an object of $\mathbb{T}$ and $M \succ L$, then L belongs to $\mathbb{T}$. One can show that, for any object M, the subcategory $[M]$ is the smallest topologizing subcategory containing M.

A topologizing category is called *thick* if it is closed under extensions.

For a thick subcategory $\mathbb{T}$ of C_X, we denote by $X/|\mathbb{T}|$ the 'space' defined by $C_{X/|\mathbb{T}|} = C_X/\mathbb{T}$. Recall that $|\mathbb{T}|$ denote the 'space' corresponding to the category $\mathbb{T}$; i.e. it is defined by $\mathbb{T} = C_{|\mathbb{T}|}$.

2.4.1. Proposition. *(a) For any topologizing subcategory $\mathbb{T}$ of C_X, there is a natural embedding $Spec(|\mathbb{T}|) \hookrightarrow Spec(X)$ which induces an embedding $\mathbf{Spec}(|\mathbb{T}|) \hookrightarrow \mathbf{Spec}(X)$.*

(b) If $\mathbb{T}$ is a thick subcategory of C_X, then there is an embedding

$$\mathbf{Spec}(X) \hookrightarrow \mathbf{Spec}(|\mathbb{T}|) \coprod \mathbf{Spec}(X/|\mathbb{T}|).$$

Sketch of proof. (a) Objects of $Spec|\mathbb{T}|$ are exactly those objects of $\mathbf{Spec}(X)$ which belong to $\mathbb{T}$.

(b) If $\mathbb{T}$ is a thick subcategory, then the localization functor $C_X \longrightarrow C_X/\mathbb{T}$ maps objects of $Spec(X)$ which do not belong to $Spec(|\mathbb{T}|)$ to objects of $Spec(X/|\mathbb{T}|)$. This map induces an injection $\mathbf{Spec}(X) - \mathbf{Spec}|\mathbb{T}| \hookrightarrow \mathbf{Spec}(X/|\mathbb{T}|)$.

For more details see [R, Ch.3]. ∎

The following fact will be refered to as the *local property* of the spectrum.

2.4.2. Proposition. *Let* $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ *be a conservative set of exact localizations. Then* $\mathbf{Spec}(X) = \bigcup_{i\in J} \mathbf{Spec}(U_i; X)$, *where* $\mathbf{Spec}(U_i; X) = \{[P] \in \mathbf{Spec}(X) \mid u_i^*(P) \neq 0\}$.

The map $[P] \longmapsto [u_i^*(P)]$ *is an embedding* $\mathbf{Spec}(U_i; X) \longrightarrow \mathbf{Spec}(U_i)$.

Proof. Let $[P] \in \mathbf{Spec}(X)$. Since the family of inverse image functors $\{u_i^* \mid i \in J\}$ is conservative, $u_i^*(P) \neq 0$ for some $i \in J$. Then $N \not\succ P$ for every $N \in ObKer(u_i^*)$, because $Ker(u_i^*)$, being a thick subcategory, contains with every object N all objects L of C_X such that $N \succ L$. Therefore $u_i^*(P) \in Spec(U_i)$. Since the functor u_i^* is exact, it respects the relation $\succ$. In particular, it maps the equivalent (with respect to $\succ$) objects to equivalent objects. Thus the element $\langle u_i^*(P)\rangle$ does not depend on the choice of an object P representing $[P]$, hence the assertion. ■

2.5. Serre subcategories. We remind the notion of a Serre subcategory of an abelian category as it is defined in [R, III.2.3.2].

Let $\mathbb{T}$ be a subcategory of C_X. We denote by $\mathbb{T}^-$ the full subcategory of C_X generated by all objects L of C_X such that any nonzero subquotient of L has a nonzero subobject which belongs to $\mathbb{T}$.

2.5.1. Proposition. *Let* $\mathbb{T}$ *be a subcategory of* C_X. *Then*
(a) The subcategory $\mathbb{T}^-$ *is thick.*
(b) $(\mathbb{T}^-)^- = \mathbb{T}^-$.
(c) $\mathbb{T} \subseteq \mathbb{T}^-$ *iff any subquotient of an object of* $\mathbb{T}$ *is isomorphic to an object of* $\mathbb{T}$.

Proof. See [R, III.2.3.2.1]. ■

2.5.2. Definition. A subcategory $\mathbb{T}$ of C_X is called a *Serre subcategory* if $\mathbb{T}^- = \mathbb{T}$.

2.5.3. Proposition. *Let* C_X *be an abelian category. For any* $P \in Spec(X)$, *the subcategory* $\langle P\rangle$ *is a Serre subcategory of the category* C_X.

Proof. (a) By 2.5.1(c), the inclusion $\langle P\rangle \subseteq \langle P\rangle^-$ is equivalent to the property: any subquotient M of any object N of the subcategory $\langle P\rangle$ belongs to $\langle P\rangle$. This property holds. In fact, if M is a subquotient of N, then $N \succ M$. If $M \notin Ob\langle P\rangle$, then, by definition of $\langle P\rangle$, $M \succ P$. Thanks to the transitivity of $\succ$, this implies that $N \succ P$ which contradicts to the assumption that $N \in Ob\langle P\rangle$.

(b) Suppose $\langle P\rangle^-$ has an object, L, which does not belong to $\langle P\rangle$. The latter means that $L \succ P$, i.e. P is a subquotient of a coproduct, $L^{\oplus n}$ of n copies of L for some finite n. Since $\langle P\rangle^-$ is closed under finite coproducts, P is a nonzero subquotient of an object of $\langle P\rangle^-$, hence it has a nonzero subobject, M, which belongs to $\langle P\rangle$. But, P belongs to $Spec(X)$, hence the object M is equivalent to P which contradicts to that $M \in Ob\langle P\rangle$. This shows that the Serre subcategory $\langle P\rangle^-$ is contained in $\langle P\rangle$. ■

2.5.3.1. Proposition. *Let* C_X *be an abelian category.*
(a) For any topologizing subcategory $\mathbb{T}$ *of the category* C_X, *the spectrum of* $\mathbb{T}$ *coincides with the spectrum of the minimal Serre subcategory,* $\mathbb{T}^-$, *containing* $\mathbb{T}$.
(b) There is a natural embedding $Spec(X) \hookrightarrow Spec(|\mathbb{T}|)\bigcup Spec(X/|\mathbb{T}^-|)$.

Proof. (a) Let $P \in Spec(\mathbb{T}^-)$. Since P is a nonzero object of $\mathbb{T}^-$, it has a nonzero subobject, P' which belong to $\mathbb{T}$. Since P is an object of the spectrum, $P' \succ P$, which implies (because $\mathbb{T}$ is topologizing) that $P \in Ob\mathbb{T}$.

(b) Since $\mathbb{T}^-$ is a thick subcategory of C_X, by 2.4.1(b), there is a natural embedding $Spec(X) \hookrightarrow Spec(|\mathbb{T}^-|) \bigcup Spec(X/|\mathbb{T}^-|)$, and by (a) above $Spec(|\mathbb{T}^-|) = Spec(|\mathbb{T}|)$, hence the assertion. ■

2.5.4. The property (sup). Recall that X (or the corresponding category C_X) has the property (sup) if for any ascending chain, Ω, of subobjects of an object M, the supremum of Ω exists, and for any subobject L of M, the natural morphism

$$sup(N \cap L \mid N \in \Omega) \longrightarrow (sup\Omega) \cap L$$

is an isomorphism.

Recall that a subcategory $\mathbb{S}$ of C_X is called *coreflective* if the inclusion functor $\mathbb{S} \hookrightarrow C_X$ has a right adjoint. In other words, the subcategory $\mathbb{S}$ is coreflective iff every object of C_X has a biggest subobject which belongs to $\mathbb{S}$.

2.5.5. Lemma. *(a) Any coreflective thick subcategory of an abelian category C_X is a Serre subcategory.*

(b) If X has the property (sup), then any Serre subcategory of C_X is coreflective.

Proof. See [R, III.2.4.4]. ■

2.5.5.1. Note. If C_X is a category with small coproducts, then a thick subcategory of C_X is coreflective iff it is closed under small coproducts (taken in C_X).

2.5.6. Proposition. *Let $\mathbb{T}$ be a full subcategory of an abelian category C_X which is closed under taking subquotients.*

(a) The smallest thick subcategory containing $\mathbb{T}$ is generated by objects M which have a finite filtration, $0 = M_0 \hookrightarrow M_1 \hookrightarrow \ \dots \ \hookrightarrow M_n = M$ such that M_{i+1}/M_i belongs to $\mathbb{T}$ for $0 \le i \ge n-1$.

(b) If X has the property (sup), then an object M of C_X belongs to $\mathbb{T}^-$ iff it has an increasing filtration $\{M_i \mid i \ge 0\}$ such that $M = sup(M_i|i \in J)$ and $M_i = sup(M_j|j < i)$ for every limit ordinal i, and M_{i+1}/M_i belongs to $\mathbb{T}$ if i is not a limit ordinal.

Proof. (a) Let $\mathbb{T}_n$ denote the full subcategory of C_X whose objects, M, have a finite filtration, $0 = M_0 \hookrightarrow M_1 \hookrightarrow \ \dots \ \hookrightarrow M_n = M$ such that M_{i+1}/M_i belongs to $\mathbb{T}$ for $0 \le i \ge n-1$. Thus, $\mathbb{T}_1 = \mathbb{T}$. It is easy to see that all $\mathbb{T}_n$ are subcategories of the smallest thick subcategory, $\mathbb{T}_\infty$, containing $\mathbb{T}$. On the other hand, if $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence such that $M' \in Ob\mathbb{T}_n$ and $M'' \in Ob\mathbb{T}_m$, then $M \in Ob\mathbb{T}_{n+m}$. Besides, every subcategory $\mathbb{T}_n$ is closed under taking subquotients. Therefore $\mathbb{T}_\infty = \bigcup_{n\ge 0} \mathbb{T}_n$.

(b) Let $\widetilde{\mathbb{T}}$ denote (temporarily) the full subcategory of C_X whose objects are supremums of their subobjects which belong to $\mathbb{T}_\infty$. The claim is that $\widetilde{\mathbb{T}}$ coincides with $\mathbb{T}^-$.

Since $\mathbb{T}$ is closed under taking subquotients, it is contained in $\mathbb{T}^-$, hence $\mathbb{T}_\infty$ is a subcategory of $\mathbb{T}^-$, Thanks to the property (sup), $\mathbb{T}^-$ is closed under supremums, which shows that $\widetilde{\mathbb{T}} \subseteq \mathbb{T}^-$.

Notice that the map assigning to every object M of the category C_X the supremum of all its subobjects which belong to $\mathbb{T}_\infty$ defines a functor which is a right adjoint to the inclusion functor $\widetilde{\mathbb{T}} \hookrightarrow C_X$. This shows that $\widetilde{\mathbb{T}}$ is coreflective. It is easy to show that it is also thick. By 2.5.5, $\widetilde{\mathbb{T}} = \widetilde{\mathbb{T}}^-$. Therefore, the inclusions $\mathbb{T} \subseteq \widetilde{\mathbb{T}} \subseteq \mathbb{T}^-$ imply that $\widetilde{\mathbb{T}} = \mathbb{T}^-$.

Thus, every object of $\mathbb{T}^-$ has an increasing filtration by objects of $\mathbb{T}_\infty$. The assertion now follows from (a). ∎

2.5.6.1. Remark. Suppose $\mathbb{T}$ is a strictly full coreflective subcategory of C_X, i.e. the inclusion functor has a right adjoint, or, equivalently, every object, M, of C_X has the $\mathbb{T}$-*torsion* which is, by definition, the biggest subobject, $\mathfrak{t}_{\mathbb{T}}(M)$, of M which belongs to $\mathbb{T}$. Then the object M has a canonical filtration defined as follows: $M_0 = 0$; if i is not a limit ordinal, then $M_{i+1}/M_i = \mathfrak{t}_{\mathbb{T}}(M/M_i)$ (in particular, $M_1 = \mathfrak{t}_{\mathbb{T}}(M)$); if i is a limit ordinal, then $M_i = sup(M_j | j < i)$. If X has the property (sup) and the subcategory $\mathbb{T}$ is closed under taking subquotients, than $sup(M_i)$ is the $\mathbb{T}^-$-torsion of M. In particular, $\mathbb{T}^-$ consists of all objects M such that $M = sup(M_i)$.

3. Local 'spaces', the S-spectrum, and the complete spectrum.

3.1. Local 'spaces' and categories. Let X be an object of $|\mathfrak{Ab}|^o$, i.e. C_X is an abelian category. A nonzero object P of the category C_X is called *quasi-final* if $N \succ P$ for every nonzero object N of C_X, or, equivalently, $\langle P \rangle = 0$.

It follows that any quasi-final object of C_X belongs to $Spec(X)$ and all quasi-final objects are equivalent to each other.

We call X (and the category C_X) *local* if C_X has a quasi-final object.

If X is local, then $\mathbf{Spec}(X)$ has a (unique) final object which coincides with the smallest topologizing subcategory of C_X. This final object is a unique closed point of the spectrum (in the topology associated with the preorder $\supseteq$).

3.1.1. Proposition. *Let X be local, and let the category C_X have simple objects. Then all simple objects of C_X are isomorphic to each other, and any quasi-final object is a direct sum of copies of a simple object.*

Proof. See [R, III.3.1.2]. ∎

3.1.2. Proposition. *For any $P \in Spec(X)$, the quotient 'space' $X/|\langle P \rangle|$ (defined by $C_{X/|\langle P \rangle|} = C_X/\langle P \rangle$) is local.*

Proof. The localization functor $q^* : C_X \longrightarrow C_{\langle P \rangle}$ is exact and its kernel coincides with $\langle P \rangle$. The latter means that every nonzero object of $C_{X/|\langle P \rangle|}$ is isomorphic to an object $q^*(N)$ such that $N \succ P$. Since q^* is exact and preserves colimits of small diagrams, it respects the preorder $\succ$. In particular, $q^*(N) \succ q^*(P)$. Since P does not belong to the subcategory $\langle P \rangle$, the object $q^*(P)$ is nonzero. All together shows that $q^*(P)$ is a quasi-final object. ∎

3.2. Residue 'space' and residue (skew) field of a local category. Let X be local, and let P be a quasi-final object of the category C_X. Since $N \succ P$ for every nonzero object of C_X, the object P belongs to all nonzero topologizing subcategories of C_X, i.e. $[P]$ is the smallest nonzero topologizing subcategory of C_X. We denote the local 'space' $|[P]|$

by $X_\star$ and call it the *residue 'space'* of X. The inclusion functor $[P] \hookrightarrow C_X$ is regarded as an inverse image functor of a morphism $X \to X_\star$.

3.2.1. The residue skew field of a local 'space'. Suppose that the category C_X has a simple object, M. We denote by k_X the ring $C_X(M,M)^o$ opposite to the ring of endomorphisms of the object M. Since M is simple, k_X is a skew field which we call the *residue skew field* of the local 'space' X. It follows from 3.1.1 that the residue skew field of X (if any) is defined uniquely up to isomorphism.

It follows that $\bar{\mathbf{x}}$ is naturally isomorphic to the (categoric) spectrum of k_X, i.e. the category $C_{\bar{\mathbf{x}}}$ is equivalent to the category $k_X - mod$ of modules (vector spaces) over k_X.

3.3. The S-spectrum. We define the *S-spectrum* of X as the preorder (with respect to $\supseteq$) of Serre subcategories, $\mathcal{P}$, of the category C_X such that the quotient 'space' $X/|\mathcal{P}|$ (or the quotient category $C_X/\mathcal{P}$) is local. We denote the S-spectrum of X by $\mathbf{Spec}^-(X)$.

The S-spectrum of X can be viewed as the preordered set of equivalence classes of exact localizations $X_\mathcal{P} \xrightarrow{q_\mathcal{P}} X$ such that the 'space' $X_\mathcal{P}$ is local and $Ker(q_\mathcal{P}^*)$ is a Serre subcategory of C_X. The preorder is given by 'specialization': a point $x' \xrightarrow{q'} X$ of the spectrum is a specialization of $x \xrightarrow{q} X$ if q' factors through q. One might regard local 'spaces' as 'fat points', and interpret exact localizations from 'fat points' to X as geometric points of X.

3.3.1. The spectrum and the S-spectrum. It follows from 2.5.3 and 3.1.2 that for any $X \in Ob|\mathfrak{Ab}|^o$, the map

$$\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^-(X), \quad [P] \longmapsto \langle P \rangle,$$

is a monomorphism of preorders which identifies $\mathbf{Spec}(X)$ with a subpreorder of $\mathbf{Spec}^-(X)$.

3.3.2. Locally noetherian 'spaces'. We call $X \in Ob|\mathfrak{Ab}|^o$ *locally noetherian* if the category C_X is a locally noetherian abelian category. It is argued in [R, Ch. VI] that if X is locally noetherian, then elements of $\mathbf{Spec}^-(X)$ are in one-to-one correspondence with the set of isomorphism classes of indecomposable injectives of the category C_X. In other words, $\mathbf{Spec}^-(X)$ is isomorphic to the Gabriel spectrum of the category C_X.

3.3.3. Remark. If X is the spectrum of a commutative noetherian ring R (i.e. $C_X = R - mod$), then the Gabriel spectrum of C_X (hence $\mathbf{Spec}^-(X)$) is isomorphic to the prime spectrum of the ring R [Gab]. If R is a non-noetherian commutative ring, $\mathbf{Spec}^-(X)$ might be much bigger than the prime spectrum of R, while $\mathbf{Spec}(X)$ is naturally isomorphic to the prime spectrum of R (cf. [R], Ch.3).

This suggests that $\mathbf{Spec}(X)$ is a better candidate for the role of the underlying space of a commutative (and a noncommutative) scheme $\mathbf{X}$. Here C_X is the category of quasi-coherent sheaves on $\mathbf{X}$. We shall see that this is the case if C_X is the category of quasi-coherent sheaves of a quasi-compact quasi-separated scheme. In the case of a general scheme, $\mathbf{Spec}(X)$ can be used as a building block for a (re)construction of the underlying space.

3.4. Functorial properties of the S-spectrum. Let $|\mathcal{L}_s\mathfrak{Ab}|^o$ denote a diagram scheme (in terminology of [GZ, Dictinary]) whose objects are $X \in Ob|\mathfrak{Ab}|^o$ with the property (sup). Morphisms of $|\mathcal{L}_s\mathfrak{Ab}|^o$ are exact localizations $X \xrightarrow{u} Y$ such that the kernel, $Ker(u^*)$, of an inverse image functor of u is a Serre subcategory of the category C_Y.

3.4.1. Lemma. *The composition of morphisms of $|\mathcal{L}_s\mathfrak{Ab}|^o$ is a morphism of $|\mathcal{L}_s\mathfrak{Ab}|^o$; i.e. $|\mathcal{L}_s\mathfrak{Ab}|^o$ is a subcategory of $|\mathfrak{Ab}|^o$.*

Proof. Let $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ be morphisms of $|\mathcal{L}_s\mathfrak{Ab}|^o$. Set $\mathbb{S} = Ker(f^*)$ and $\mathbb{T} = Ker(g^*)$. Let M be an object of C_Z which does not belong to the kernel of $C_Z \xrightarrow{f^*g^*} C_X$, i.e. $g^*(M)$ does not belong to the Serre subcategory $\mathbb{S}$ of C_Y. Since Y has the property (sup), the category $\mathbb{S}$ is coreflective (cf. 2.5.5). Denote by $M_{\mathbb{S}}$ the quotient of the object M by its $\mathbb{S}$-torsion and by ϕ the canonical epimorphism $g^*(M) \longrightarrow M_{\mathbb{S}}$. Since g^* is a localization, there exists a diagram $M \xleftarrow{s} K' \xrightarrow{\phi'} M'$ such that $g^*(s)$ is an isomorphism, ϕ' is an epimorphism, and there exists an isomorphism $g^*(M') \xrightarrow{t} M_{\mathbb{S}}$ making the diagram

$$\begin{array}{ccc} g^*(M) & \xrightarrow{\phi} & M_{\mathbb{S}} \\ {\scriptstyle g^*(s)}\uparrow & & \uparrow{\scriptstyle t} \\ g^*(K') & \xrightarrow{\phi'} & g^*(M') \end{array}$$

commute. Replacing K' by the image, K, of s and M' by the coproduct $M'' = K \coprod_{K'} M'$, we can assume that s is a monomorphism, i.e. M' is a subquotient of the object M. Finally, replacing M' by the quotient of M' by its $\mathbb{T}$-torsion (which is possible due to the coreflectivity of $\mathbb{T}$), we obtain a nonzero subquotient of M which is both $\mathbb{S}$- and $\mathbb{T}$-torsion free. Therefore it cannot have a nonzero subobject which is $Ker(f^*g^*)$-free. This shows that $M \notin Ker(f^*g^*)^-$. Therefore $Ker(f^*g^*)^- \subseteq Ker(f^*g^*)$, i.e. $Ker(f^*g^*) = Ker(f^*g^*)^-$. ∎

3.4.2. Proposition. *Any morphism $Y \xrightarrow{\phi} X$ of the category $|\mathcal{L}_s\mathfrak{Ab}|^o$ induces an injective map of spectra $\mathbf{Spec}^-(Y) \xrightarrow{\phi^a} \mathbf{Spec}^-(X)$. The correspondence*

$$X \longmapsto \mathbf{Spec}^-(X), \quad \phi \mapsto \phi^a, \tag{1}$$

is functorial.

Proof. The map $\mathbf{Spec}^-(Y) \xrightarrow{\phi^a} \mathbf{Spec}^-(X)$ is given by $q \longmapsto \phi \circ q$. The functoriality of the map (1) is immediate. ∎

3.4.3. Remark. Let $U \xrightarrow{u} X$ be a morphism of $|\mathcal{L}_s\mathfrak{Ab}|^o$; i.e. u^* is a localization whose kernel is a Serre subcategory of C_X. By 3.4.2 and 3.3.1, we have canonical injective maps $\mathbf{Spec}(U) \longrightarrow \mathbf{Spec}^-(U) \longrightarrow \mathbf{Spec}^-(X)$. This allows to identify $\mathbf{Spec}(U)$ with a subset of $\mathbf{Spec}^-(X)$.

3.5. The complete spectrum. The *complete spectrum* of X is a preorder (with resp. to $\supseteq$) of all thick subcategories $\mathcal{P}$ such that the quotient 'space' $X/|\mathcal{P}|$ is local. We denote it by $\mathbf{Spec}^1(X)$. Obviously, $\mathbf{Spec}^-(X) \subseteq \mathbf{Spec}^1(X)$. It follows that the map $X \longmapsto \mathbf{Spec}^1(X)$ is functorial with respect to exact localizations. Thus, for any exact localization $U \xrightarrow{u} X$, we have canonical embeddings

$$\mathbf{Spec}(U) \hookrightarrow \mathbf{Spec}^-(U) \hookrightarrow \mathbf{Spec}^1(U) \hookrightarrow \mathbf{Spec}^1(X)$$

which allow to identify $\mathbf{Spec}(U)$ with a subset of $\mathbf{Spec}^1(X)$. In terms of this identification, the local property of the spectrum (2.4.2) can be reformulated as follows:

For any set $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ of exact localizations, $\mathbf{Spec}(X) \hookrightarrow \bigcup_{i\in J} \mathbf{Spec}(U_i)$.

3.5.1. Note. The complete spectrum was first introduced in [R, Ch.VI], together with the S-spectrum. The S-spectrum of a 'space' X can be viewed as the smallest extension of the spectrum $\mathbf{Spec}(X)$ containing $\mathbf{Spec}(U)$ for all Serre localizations $U \hookrightarrow X$. Similarly, the complete spectrum of a 'space' X is the smallest extension of $\mathbf{Spec}(X)$ containing $\mathbf{Spec}(U)$ for all exact localizations $U \hookrightarrow X$. If C_X is a category with colimits (say, a Grothendieck category), then Serre localizations are exact localizations with a direct image functor. Even if X corresponds to a commutative scheme (i.e. C_X is the category of quasi-coherent sheaves on a scheme) and $U \xrightarrow{u} X$ corresponds to an open immersion, the existence of a direct image functor is a certain finiteness condition.

4. The pretopology of Serre localizations and the S-spectrum.

4.1. Lemma. *Let C_X be an abelian category. For any finite set $\{T_i \mid i \in J\}$ of topologizing subcategories of C_X, we have the equality* $\big(\bigcap_{i\in J} T_i\big)^- = \bigcap_{i\in J} T_i^-$.

Proof. Clearly $\big(\bigcap_{i\in J} T_i\big)^- \subseteq \bigcap_{i\in J} T_i^-$. We need to prove the inverse inclusion.

Let $J = \{1, 2, ..., n\}$. Let M be a nonzero object of $\bigcap_{i\in J} T_i^-$. And let L be any nonzero subquotient of the object M. Since M is a nonzero object of T_1^-, the object L has a nonzero subobject, L_1, which belongs to T_1. Since M is a nonzero object of T_2^- and L_1 is a nonzero subquotient of M, the object L_1 has a nonzero subobject, L_2, which belongs to T_2. Since T_1 contains all subobjects of its objects, $L_2 \in T_1 \bigcap T_2$. Continuing this way, we obtain a descending chain of nonzero subobjects of L, $L_n \to L_{n-1} \to ... \to L_1 \to L$, such that $L_i \in Ob \bigcap_{1\le j\le i} T_j$. This shows that $M \in \big(\bigcap_{i\in J} T_i\big)^-$. ■

4.2. The Gabriel multiplication. The *Gabriel product* of two subcategories $\mathbb{T}$ and $\mathbb{S}$ of an abelian category C_X is the full subcategory $\mathbb{T}\bullet\mathbb{S}$ of C_X generated by all $M \in ObC_X$ for which there exists an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

with $M' \in Ob\mathbb{S}$ and $M'' \in Ob\mathbb{T}$. If $\mathbb{T}$ and $\mathbb{S}$ are topologizing subcategories, then such is $\mathbb{T} \bullet \mathbb{S}$. This multiplication is associative and has an identity element – the subcategory $\mathbf{0}$. Note that a topologizing subcategory $\mathbb{T}$ of C_X is *thick* iff $\mathbb{T} \bullet \mathbb{T} = \mathbb{T}$.

By Lemma III.6.2.1 in [R], if $\mathbb{T}$, $\mathbb{S}$ are coreflective subcategories of C_X, their Gabriel product $\mathbb{T} \bullet \mathbb{S}$ is coreflective too.

4.2.1. Lemma. *Let C_X be an abelian category. For any finite set $\{S,\ T_i \mid i \in J\}$ of topologizing subcategories of the category C_X, the following equalities hold:*

$$\Big(\bigcap_{i\in J} T_i\Big) \bullet S = \bigcap_{i\in J} T_i \bullet S \quad \textit{and} \quad S \bullet \Big(\bigcap_{i\in J} T_i\Big) = \bigcap_{i\in J} S \bullet T_i.$$

Proof. (a) The inclusions $\big(\bigcap_{i\in J} T_i\big)\bullet S \subseteq \bigcap_{i\in J} T_i\bullet S$ and $S\bullet\big(\bigcap_{i\in J} T_i\big)\subseteq \bigcap_{i\in J} S\bullet T_i$ are evident. We need to prove the inverse inclusions.

(b) Let $M\in Ob\bigcap_{i\in J} T_i\bullet S$; i.e. for any $i\in J$, there is a monomorphism $f_i : M_i\to M$ such that $M_i\in ObT_i$ and $Cok(f_i)\in ObS$. This gives an exact sequence

$$0\longrightarrow \bigcap_{i\in J} M_i\longrightarrow M\longrightarrow \prod_{i\in J} Cok(f_i)$$

Clearly $\bigcap_{i\in J} M_i\in Ob\bigcap_{i\in J} T_i$. Since J is finite, $\prod_{i\in J} Cok(f_i)\in ObS$, hence $M\in Ob\big(\bigcap_{i\in J} T_i\big)\bullet S$.

(c) Suppose $S\bullet M\in Ob\bigcap_{i\in J} T_i$; i.e. for any $i\in J$, there is a monomorphism $M_i\xrightarrow{f_i} M$ such that $M_i\in ObS$ and $Cok(f_i)\in ObT_i$. Since J is finite, $sup(M_i|i\in J)\in ObS$, and $M/sup(M_i|i\in J)\in Ob\bigcap_{i\in J} T_i$, hence $M\in\bigcap_{i\in J} S\bullet T_i$. ■

For any pair S and T of Serre subcategories of the category C_X, the symbol $S\vee T$ denotes the minimal Serre subcategory of C_X containing S and T. It follows that $S\vee T = (S\bullet T)^-$.

4.3. Proposition. *Let C_X be an abelian category. For any finite set $\{T_i|i\in J\}$ of Serre subcategories of the category C_X, the equality $\big(\bigcap_{i\in J} T_i\big)\vee S = \bigcap_{i\in J}(T_i\vee S)$ holds.*

Proof. By Lemma 4.1, $\bigcap_{i\in J}(T_i\vee S) = \bigcap_{i\in J}(T_i\bullet S)^- = \big(\bigcap_{i\in J}(T_i\bullet S)\big)^-$. By Lemma 4.2.1, $\big(\bigcap_{i\in J}(T_i\bullet S)\big)^- = \big((\bigcap_{i\in J} T_i)\bullet S\big)^- = \big(\bigcap_{i\in J} T_i\big)\vee S$. ■

4.4. The pretopology of Serre localizations. We define the *quasi-pretopology of Serre localizations*, $\tau_{\mathfrak{L}_\mathfrak{s}}$, on the category $|\mathfrak{L}_\mathfrak{s}\mathfrak{Ab}|^o$ by taking as covers all families of morphisms $\{U_i\stackrel{u_i}{\to} X\mid i\in J\}$ such that the corresponding family of inverse image functors is conservative. We define by $\tau^{\mathfrak{f}}_{\mathfrak{L}_\mathfrak{s}}$ the quasi-pretopology on $|\mathfrak{L}_\mathfrak{s}\mathfrak{Ab}|^o$ obtained by taking all covers of $\tau_{\mathfrak{L}_s}$ containing a finite subcover.

It follows from Proposition 4.3 that $\tau^{\mathfrak{f}}_{\mathfrak{L}_s}$ is a Grothendieck pretopology. We call it the *pretopology of Serre localizations.*

4.5. The local property of the S-spectrum.

4.5.1. Proposition. *Let $\{U_i\stackrel{u_i}{\to} X\mid i\in J\}$ be a cover in the pretopology $\tau^{\mathfrak{f}}_{\mathfrak{L}_s}$. Then* $\mathbf{Spec}^-(X) = \bigcup_{i\in J}\mathbf{Spec}^-(U_i)$.

Proof. The inclusion $\bigcup_{i\in J}\mathbf{Spec}^-(U_i) \subseteq \mathbf{Spec}^-(X)$ follows from the functoriality of the S-spectrum (see 3.4.1). We need to check the inverse inclusion.

Let $\mathbf{x} \xrightarrow{q} X$ be any point of $\mathbf{Spec}^-(X)$. Consider cartesian squares

$$\begin{array}{ccc} U_i^{\mathbf{x}} & \xrightarrow{q_i} & U_i \\ {\scriptstyle u_i^{\mathbf{x}}}\downarrow & & \downarrow{\scriptstyle u_i} \\ \mathbf{x} & \xrightarrow{q} & X \end{array} \qquad i \in J \tag{1}$$

Since $\tau^{\mathfrak{f}}_{\mathfrak{L}_s}$ is a pretopology, the pull-back $\{U_i^{\mathbf{x}} \xrightarrow{u_i^{\mathbf{x}}} \mathbf{x} \mid i \in J\}$ of the cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is a cover. Let P be a quasi-final object of the local category $C_{\mathbf{x}}$. Since (by the definition of a cover) the set of inverse image functors $\{C_{\mathbf{x}} \xrightarrow{u_i^{\mathbf{x}*}} C_{U_i^{\mathbf{x}}} \mid i \in J\}$ is conservative, $u_j^{\mathbf{x}*}(P) \neq 0$ for some $j \in J$. Since P is the point of the spectrum of $\mathbf{x}$, the object $u_j^{\mathbf{x}*}(P)$ belongs to the spectrum, hence to the flat spectrum, of $U_j^{\mathbf{x}}$. By functoriality of the S-spectrum (cf. 3.4.1), the morphism $U_j^{\mathbf{x}} \xrightarrow{q_j} U_j$ induces a map $\mathbf{Spec}^-(U_j)^{\mathbf{x}} \longrightarrow \mathbf{Spec}^-(U_j)$ which sends the point $\langle u_j^{\mathbf{x}*}(P)\rangle$ to a point $\mathbf{P}_j$ of $\mathbf{Spec}^-(U_j)$. It follows from the commutativity of (1) that the image of $\mathbf{P}_j$ by $U_j \xrightarrow{u_j} X$ coincides with $\mathbf{x} \xrightarrow{q} X$. ∎

4.5.2. Corollary. *Let $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a finite conservative set of exact localizations. Then $\mathbf{Spec}^-(X) \subseteq \bigcup_{i\in J}\mathbf{Spec}^-(U_i)$. Here $\mathbf{Spec}^-(X)$ and $\mathbf{Spec}^-(U_i)$, $i \in J$, are realized as subsets of the complete spectrum $\mathbf{Spec}^1(X)$.*

Proof. Let T_i denote the kernel of the localization functor $C_X \xrightarrow{u_i^*} C_{U_i}$, $i \in J$. The condition that $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is conservative means that $\bigcup_{i\in J} T_i = 0$. By 4.1, $\bigcap_{i\in J} T_i^- = \big(\bigcap_{i\in J} T_i\big)^- = 0$, or, equivalently, $\{\widetilde{U}_i \xrightarrow{\widetilde{u}_i} X \mid i \in J\}$ is a finite cover of X by Serre localizations. Here $\widetilde{U}_i$ denote the 'space' $X/|T_i^-|$ (i.e. $C_{\widetilde{U}_i} = C_X/T_i^-$). Since $T_i \subseteq T_i^-$, the localization $\widetilde{U}_i \longrightarrow X$ factors through a localization $\widetilde{U}_i \longrightarrow U_i$ with the kernel T_i^-/T_i. The latter is a Serre subcategory of the quotient category C_X/T_i. Therefore, $\mathbf{Spec}^-(\widetilde{U}_i) \subseteq \mathbf{Spec}^-(U_i)$. By 4.5.1, $\mathbf{Spec}^-(X) = \bigcup_{i\in J}\mathbf{Spec}^-(\widetilde{U}_i) \subseteq \bigcup_{i\in J}\mathbf{Spec}^-(U_i)$. ∎

4.6. The pretopology of exact localizations and the complete spectrum. There are similar facts for exact localizations and complete spectrum.

For any pair S and T of Serre subcategories of the category C_X, the symbol $S \sqcup T$ denotes the minimal thick subcategory of C_X containing S and T.

4.6.1. Proposition. *Let C_X be an abelian category. Let $\{T_i | i \in J\}$ be a finite family of thick subcategories of the category C_X. Then $\big(\bigcap_{i\in J} T_i\big) \sqcup S = \bigcap_{i\in J}(T_i \sqcup S)$ for any thick subcategory S.*

Proof. The inclusion $\big(\bigcap_{i\in J} T_i\big) \sqcup S \subseteq \bigcap_{i\in J}(T_i \sqcup S)$ is evident. We need to prove the inverse inclusion. Let T denote the intersection $\bigcup_{i\in J} T_i$. Replacing S by $S \sqcup T$ and C_X by the quotient category C_X/T, we can assume that $T = 0$. Then by 4.3, we have the inclusions

$$S = \big(\bigcap_{i\in J} T_i\big) \sqcup S \subseteq \bigcap_{i\in J}(T_i \sqcup S) \subseteq \bigcap_{i\in J}(T_i \vee S) = \big(\bigcap_{i\in J} T_i\big) \vee S = S,$$

hence the assertion. ■

4.6.2. The pretopology of exact localizations. Let $|\mathfrak{L}_{\mathfrak{e}}\mathfrak{Ab}|^o$ be a category whose objects are 'spaces' X with abelian category C_X and morphisms are exact localizations. We define the *quasi-pretopology of exact localizations*, $\tau_{\mathfrak{L}_{\mathfrak{e}}}$, on the category $|\mathfrak{L}_{\mathfrak{e}}\mathfrak{Ab}|^o$ by taking as covers all families of morphisms $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ such that the corresponding family of inverse image functors is conservative. We define by $\tau^{\mathfrak{f}}_{\mathfrak{L}_{\mathfrak{e}}}$ the quasi-pretopology on $|\mathfrak{L}_{\mathfrak{e}}\mathfrak{Ab}|^o$ obtained by taking all covers of $\tau_{\mathfrak{L}_e}$ containing a finite subcover.

It follows from Proposition 4.6.1 that $\tau^{\mathfrak{f}}_{\mathfrak{L}_e}$ is a Grothendieck pretopology. We call it the *pretopology of exact localizations*, or simply the *pretopology of localizations*.

The following assertion is refered to as the *local property* of the complete spectrum.

4.6.3. Proposition. *Let $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a cover in the pretopology $\tau^{\mathfrak{f}}_{\mathfrak{L}_e}$. Then* $\mathbf{Spec}^1(X) = \bigcup_{i\in J} \mathbf{Spec}^1(U_i)$.

Proof. The argument is similar to that of 4.5.1. Details are left to the reader. ■

5. The upper spectrum of a quasi-topological 'space'.

We fix a quasi-presite $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a subcategory of $|\mathfrak{Ab}|^o$ and τ-covers are sets of morphisms $\{U_i \xrightarrow{u_i} Y \mid i \in J\}$ such that $\{C_Y \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ is a conservative family of exact localizations.

For any $X \in Ob\mathcal{A}$, we define the *upper spectrum*, $\mathbf{Spec}(X, \tau_X)$, of the quasi-topological 'space' (X, τ_X) as the (preordered) subset of the complete spectrum $\mathbf{Spec}^1(X)$ formed by all elements $\mathcal{P}$ such that the embedding of the quotient local 'space' $X/|\mathcal{P}| \to X$ factors through an embedding $\mathbf{Spec}(U) \hookrightarrow \mathbf{Spec}^1(X)$ for some element of a cover $U \xrightarrow{u} X$. In other words, $Ker(u^*) \subseteq \mathcal{P}$ and $\mathcal{P}/Ker(u^*) = \langle M \rangle$ for some object M of $Spec(U)$. It follows from this definition that

$$\mathbf{Spec}(X, \tau_X) = \bigcup_{(U\to X)\in\Sigma^\tau X} \mathbf{Spec}(U).$$

Here $\mathbf{Spec}(U)$ is identified with its image in $\mathbf{Spec}^1(X)$. Since Σ^τ is closed under composition, the map $X \longmapsto \mathbf{Spec}(X, \tau_X)$ is functorial with respect to Σ^τ.

5.1. Observations. (a) If $\mathcal{A}$ coincides with the category $|\mathfrak{L}_{\mathfrak{e}}\mathfrak{Ab}|^o$ and τ is a discrete pretopology (i.e. all covers are isomorphisms), then $\mathbf{Spec}(X, \tau_X) = \mathbf{Spec}(X)$.

(b) If $\mathcal{A}$ is the category $|\mathfrak{L}_{\mathfrak{s}}\mathfrak{Ab}|^o$ and τ is the quasi-pretopology of Serre localizations, $\tau_{\mathfrak{L}_s}$, or its finite version, the pretopology $\tau^{\mathfrak{f}}_{\mathfrak{L}_s}$ (cf. 4.3, 4.4), then $\mathbf{Spec}(X,\tau_X)$ coincides with the S-spectrum: $\mathbf{Spec}(X,\tau_X) = \mathbf{Spec}^-(X)$.

(c) If $\mathcal{A}$ is the category $|\mathfrak{L}_{\mathfrak{s}}\mathfrak{Ab}|^o$ and τ is the quasi-pretopology of exact localizations, $\tau_{\mathfrak{L}_{\mathfrak{e}}}$, or its finite version, the pretopology $\tau^{\mathfrak{f}}_{\mathfrak{L}_{\mathfrak{e}}}$ (cf. 4.3, 4.4), then $\mathbf{Spec}(X,\tau_X)$ coincides with the complete spectrum: $\mathbf{Spec}(X,\tau_X) = \mathbf{Spec}^1(X)$.

(d) In general, there are inclusions

$$\mathbf{Spec}(X) \subseteq \mathbf{Spec}(X,\tau_X) \subseteq \mathbf{Spec}^1(X).$$

If all elements of τ_X-covers are Serre localizations, than

$$\mathbf{Spec}(X) \subseteq \mathbf{Spec}(X,\tau_X) \subseteq \mathbf{Spec}^-(X).$$

Due to (b) above, the following fact might be regarded as an extension to the general case of the 'local property' of the S-spectrum (see 4.5.1).

5.2. Proposition. *Suppose* $\Sigma^\tau X \subseteq \Sigma_\tau X$. *Then* $\mathbf{Spec}(X,\tau_X) = \bigcup_{i\in J} \mathbf{Spec}(U_i,\tau_{U_i})$ *for any* τ*-cover* $\{U_i \to X \mid i \in J\}$.

Proof. The inclusion $\bigcup_{i\in J} \mathbf{Spec}(U_i,\tau_{U_i}) \subseteq \mathbf{Spec}(X,\tau_X)$ follows from the definition of $\mathbf{Spec}(X,\tau_X)$ and does not require any additional assumptions.

Let $\mathbf{P}$ be an arbitrary element of $\mathbf{Spec}(X,\tau_X)$ and $V \to X$ a morphism of $\Sigma^\tau X$ such that $\mathbf{P} \in \mathbf{Spec}(V,\tau_V)$. Let $\{\widetilde{U}_i \to V \mid i \in J\}$ be a pull-back of the cover $\{U_i \to X \mid i \in J\}$ along the morphism $V \to X$. The condition $\Sigma^\tau \subseteq \Sigma_\tau$ means that $\{\widetilde{U}_i \to V \mid i \in J\}$ is a cover. In particular, the set of inverse image functors of morphisms of the cover is conservative. Therefore, there exists $i \in J$ such that $\mathbf{P} \in \mathbf{Spec}(\widetilde{U}_i)$ (see 2.4.4). Since the projection $\widetilde{U}_i \to U_i$ is an element of a cover (belongs to Σ^τ, this implies that $\mathbf{P} \in \mathbf{Spec}(U_i,\tau_{U_i})$. ■

5.3. A natural topology on the upper spectrum. We define a topology, τ^X, on $\mathbf{Spec}(X,\tau_X)$ by taking $\mathfrak{B}_\tau X = \{\mathbf{Spec}(U,\tau_U) \subseteq \mathbf{Spec}(X,\tau_X) \mid (U \to X) \in \Sigma^\tau X\}$ as a base of its open sets.

5.3.1. Lemma. *Suppose that* $\Sigma^\tau X \subseteq \Sigma_\tau$. *Then the intersection of any pair of sets of the base* $\mathfrak{B}_\tau X$ *belongs to* $\mathfrak{B}_\tau X$.

Proof. The condition $\Sigma^\tau X \subseteq \Sigma_\tau X$ implies that for any pair $U \longrightarrow X \longleftarrow V$ of morphisms of $\Sigma^\tau X$, the other two morphisms (projections) of the cartesian square

$$\begin{array}{ccc} U \times_X V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}$$

and the composition $U \times_X V \longrightarrow X$ belong to Σ^τ. The fact follows now from the equality

$$\mathbf{Spec}(U \times_X V, \tau_{U\times_X V}) = \mathbf{Spec}(U, \tau_U) \bigcap \mathbf{Spec}(V, \tau_V).$$

The inclusion $\mathbf{Spec}(U \times_X V, \tau_{U\times_X V}) \subseteq \mathbf{Spec}(U, \tau_U) \bigcap \mathbf{Spec}(V, \tau_V)$ is evident. The inverse inclusion follows from the definition of the spectrum and the universal property of fibered products. Details are left to the reader. ■

5.4. The upper spectrum of covers. Let $\mathcal{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a set of exact localizations. We call $\mathbf{Spec}^\dagger\mathcal{U} = \bigcup_{i\in J} \mathbf{Spec}(U_i)$ the *upper spectrum* of the family $\mathcal{U}$. If follows from 2.4.2 that if a cover $\mathcal{V}$ of X is a refinement of the cover $\mathcal{U}$, then there is a natural injective map $\mathbf{Spec}^\dagger\mathcal{U} \longrightarrow \mathbf{Spec}^\dagger\mathcal{V}$. It follows that $\mathbf{Spec}(X, \tau_X)$ is isomorphic to the colimit of $\mathbf{Spec}^\dagger\mathcal{U}$, where $\mathcal{U}$ runs through the set of all τ-covers of X.

6. The combinatorial spectrum.

6.1. The lower spectrum of a cover and of a quasi-topological 'space'. Fix a quasi-presite $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a subcategory of the category $|\mathfrak{Ab}|^o$. We assume that all τ-covers are conservative families $\{U_j \xrightarrow{u_j} X \mid j \in J\}$ of exact localizations; i.e. all inverse image functors $C_X \xrightarrow{u_j^*} C_{U_j}$ are exact localizations and the family of functors $\{u_j^* \mid j \in J\}$ is conservative.

For any cover $\mathcal{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$, we define the *lower spectrum* of $\mathcal{U}$ as the set $\mathbf{Spec}_\wp\mathcal{U}$ of all $\mathcal{P} \in \mathbf{Spec}^1(X)$ such that if $\mathcal{P} \in \mathbf{Spec}^1(U_j)$ (i.e. $Ker(u_j^*) \subseteq \mathcal{P}$) for some $j \in J$, then $\mathcal{P} \in \mathbf{Spec}(U_j)$. The latter means that there exists an object L of C_X such that $\mathcal{P}/Ker(u_j^*) = \langle u_j^*(L)\rangle$.

If follows from 2.4.2 that if a cover $\mathcal{V}$ of X is a refinement of the cover $\mathcal{U}$, then there is a natural injective map $\mathbf{Spec}_\wp\mathcal{U} \longrightarrow \mathbf{Spec}_\wp\mathcal{V}$. We denote by $\mathbf{Spec}_\wp(X, \tau_X)$ the colimit of $\mathbf{Spec}_\wp\mathcal{U}$, where $\mathcal{U}$ runs through the set $Cov_\tau(X)$ of all τ-covers of X.

For any cover $\mathcal{U}$ of X, we have $\mathbf{Spec}(X) \subseteq \mathbf{Spec}_\wp\mathcal{U} \subseteq \mathbf{Spec}(X, \tau_X)$, which imply the inclusions

$$\mathbf{Spec}(X) \subseteq \mathbf{Spec}_\wp(X, \tau_X) \subseteq \mathbf{Spec}(X, \tau_X).$$

Let τ^X denote the topology on $\mathbf{Spec}_\wp(X, \tau_X)$ induced by the natural topology on $\mathbf{Spec}(X, \tau_X)$ (cf. 5.3). We call the topological space $(\mathbf{Spec}_\wp(X, \tau_X), \tau^X)$ the *combinatorial τ-spectrum* of X.

6.1.1. A remark on spectra of a commutative scheme. For an arbitrary ringed topological space $\mathbf{X} = (\mathcal{X}, \mathcal{O})$, we denote by $Sp\mathbf{X}$ the *categoric spectrum* of $\mathbf{X}$ defined by taking as $C_{Sp\mathbf{X}}$ the category $Qcoh_{\mathbf{X}}$ of quasi-coherent sheaves on $\mathbf{X}$. Let $\mathcal{A}$ be the category whose objects are categoric spectra of commutative schemes and morphisms are morphisms of spectra corresponding to open immersions. Let τ be the pretopology on $\mathcal{A}$ corresponding to the Zariski pretopology on the category of schemes. Then for any object $X = Sp(\mathcal{X}, \mathcal{O})$ of the category $\mathcal{A}$, the combinatorial spectrum $\mathbf{Spec}_\wp(X, \tau_X)$ and the spectrum $\mathbf{Spec}(X, \tau_X)$ of (X, τ_X) both coincide with the underlying space $\mathcal{X}$ of the scheme $\mathbf{X}$. But, according to an observation by O. Gabber, the spectrum $\mathbf{Spec}(X)$ might differ from the space $\mathcal{X}$.

6.1.2. The combinatorial S-spectrum. For any cover $\mathcal{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ of (X, τ_X), we define the *lower S-spectrum*, $\mathbf{Spec}^-_\wp \mathcal{U}$, of the cover $\mathcal{U}$ as the intersection of $\mathbf{Spec}_\wp \mathcal{U}$ and the S-spectrum of X. In other words, $\mathbf{Spec}^-_\wp \mathcal{U}$ is the set of all $\mathcal{P} \in \mathbf{Spec}^-(X)$ such that if $\mathcal{P} \in \mathbf{Spec}^1(U_j)$ for some $j \in J$, then $\mathcal{P} \in \mathbf{Spec}(U_j)$.

Let $\mathbf{Spec}^-_\wp(X, \tau_X)$ denote the intersection of the combinatorial spectrum of (X, τ_X) with the S-spectrum of X and call it the *combinatorial S-spectrum* of (X, τ_X).

We show below, among other facts, that the natural map $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^-_\wp(X, \tau_X)$ is bijective, if (X, τ_X) is quasi-compact (i.e. every τ-cover of X contains a finite subcover).

One of the consequences of this fact is that if C_X is the category of quasi-coherent sheaves on a scheme $\mathbf{X}$, then $\mathbf{Spec}(X)$ is isomorphic to the set of points of the underlying topological space of $\mathbf{X}$, provided that the scheme $\mathbf{X}$ is quasi-compact and quasi-separated, or its underlying space is noetherian.

We need some preparation.

6.2. Proposition. *Let $\mathcal{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a finite set of exact localizations such that the family of inverse image functors $\{u_i^* \mid i \in J\}$ is conservative. Let M, N be objects of C_X such that N is $Ker(u_i^*)$-torsion free. Then $M \succ N$ iff $u_i^*(M) \succ u_i^*(N)$ for all $i \in J$.*

Proof. The implication $\big[M \succ N\big] \Rightarrow \big[u_i^*(M) \succ u_i^*(N)$ for all $i \in J\big]$ holds without any additional conditions on N, because all functors u_i^* are exact, hence they respect the relation $\succ$. The rest of the argument shows that the inverse implication holds provided the object N is $Ker(u_i^*)$-torsion free for all $i \in J$.

(a) The relation $u_i^*(M) \succ u_i^*(N)$ means that there exists a diagram

$$u_i^*(M)^{\oplus J_i} \xleftarrow{\mathfrak{j}_i} \widetilde{K}_i \xrightarrow{\mathfrak{e}_i} u_i^*(N) \qquad (1)$$

in which $\mathfrak{j}_i$ is a monomorphism, $\mathfrak{e}_i$ is an epimorphism, and J_i is finite. Since u_i^* is a localization functor, we can assume that $\widetilde{K}_i = u_i^*(K_i)$ for some $K_i \in ObC_X$. To the diagram (1) there corresponds a diagram

$$M^{\oplus J_i} \xrightarrow{s} M_i' \xleftarrow{\mathfrak{j}_i'} K_i \xrightarrow{\mathfrak{e}_i'} N_i \xleftarrow{t} N \qquad (2)$$

such that the morphisms $u_i^*(s)$ and $u_i^*(t)$ are invertible, and the arrows of (1) are described by $\mathfrak{j}_i = u_i^*(s)^{-1}u_i^*(\mathfrak{j}_i')$ and $\mathfrak{e}_i = u_i^*(t)^{-1}u_i^*(\mathfrak{e}_i')$. Here we identify $u_i^*(M)^{\oplus J_i}$ with $u_i^*(M^{\oplus J_i})$.

Since $\mathfrak{j}_i$ is a monomorphism, the $Ker(\mathfrak{j}_i')$ belongs to the kernel of the functor u_i^*. Replacing K_i by the image of the morphism $\mathfrak{j}_i'$ and N_i by the cokernel of the composition $Ker(\mathfrak{j}_i') \longrightarrow K_i \xrightarrow{\mathfrak{e}_i'} N_i$, we can (and will) assume that $M_i' \xleftarrow{\mathfrak{j}_i'} K_i$ is a monomorphism. This implies that the projection $K_i' = K_i \times_{M_i'} (M^{\oplus J_i}) \longrightarrow M^{\oplus J_i}$ is a monomorphism.

(b) Consider now the cartesian square

$$\begin{array}{ccc} K_i'' & \xrightarrow{\mathfrak{e}_i''} & N \\ {\scriptstyle \mathfrak{j}_i''}\downarrow & & \downarrow{\scriptstyle t} \\ K_i' & \longrightarrow & N_i \end{array}$$

in which the lower horizontal arrow is the composition of the projection $K'_i \longrightarrow K_i$ and the morphism $K_i \xrightarrow{\mathfrak{e}'_i} N_i$. Since $u_i^*(t)$ is invertible, the kernel of the morphism t belongs to $Ker(u_i^*)$. By hypothesis, N is $Ker(u_i^*)$-torsion free. Therefore t is a monomorphism. This implies that $K''_i \xrightarrow{\mathfrak{j}''_i} K'_i$ is a monomorphism, hence the composition of $K'_i \xrightarrow{\mathfrak{j}''_i} K'_i$ and $K'_i \longrightarrow M^{\oplus J_i}$ is a monomorphism. Thus, we have obtained a diagram

$$M^{\oplus J_i} \xleftarrow{\widetilde{\mathfrak{j}}_i} K''_i \xrightarrow{\widetilde{\mathfrak{e}}_i} N \tag{3}$$

whose left arrow is a monomorphism and such that its image by the localization functor u_i^* is isomorphic to the diagram (1).

(c) Let I denote the disjoint union of J_i, $i \in J$, and let $K = \oplus_{i\in J} K''_i$. Monomorphisms $\{\widetilde{\mathfrak{j}}_i \mid i \in J\}$ determine a monomorphism $K \xrightarrow{\mathfrak{j}} M^{\oplus I}$. The morphisms $\widetilde{\mathfrak{e}}_i$, $i \in J$, determine a morphism $K \xrightarrow{\mathfrak{e}} N$. We claim that $\mathfrak{e}$ is an epimorphism.

Since the set of functors $\{u_i^* \mid i \in J\}$ is conservative, in particular it reflects epimorphisms, it suffices to show that $u_i^*(\widetilde{K}) \xrightarrow{u_i^*(\mathfrak{e})} u_i^*(N)$ is an epimorphism for every $i \in J$. This is, indeed, the case, because the composition of $u_i^*(\mathfrak{e})$ with the natural morphism $u_i^*(K''_i) \longrightarrow u_i^*(K)$ coincides with $u_i^*(\widetilde{\mathfrak{e}}_i)$, and the latter is an epimorphism, as it is argued above. ■

6.2.1. Corollary. *Let $\mathcal{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a finite set of exact localizations such that the family of inverse image functors $\{u_i^* \mid i \in J\}$ is conservative. Let P be an object of C_X such that P is $Ker(u_i^*)$-torsion free for all $i \in J$. Then $P \in Spec(X)$ iff $u_i^*(P) \in Spec(U_i)$ for every $i \in J$.*

Proof. (a) Suppose $P \in Spec(X)$. Since P is $Ker(u_i^*)$-torsion free for all $i \in J$, the object $u_i^*(P)$ is nonzero for every $i \in J$. Thus $u_i^*(P) \in Spec(U_i)$ for every $i \in J$.

(b) Conversely, suppose that $u_i^*(P) \in Spec(U_i)$ for all $i \in J$. Let $M \longrightarrow P$ be a nonzero monomorphism. Since P is $Ker(u_i^*)$-torsion free for all $i \in J$, the object M has the same property. In particular, $u_i^*(M) \longrightarrow u_i^*(P)$ is a nonzero monomorphism for every $i \in J$. Since $u_i^*(P) \in Spec(U_i)$, $u_i^*(M) \succ u_i^*(P)$ for all $i \in J$. By 6.2, this implies that $M \succ P$. Therefore $P \in Spec(X)$. ■

6.3. Proposition. *Suppose C_X is an abelian category with the property (sup). Let $\mathcal{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a finite set of exact localizations such that the family of inverse image functors $\{u_i^* \mid i \in J\}$ is conservative. Then the canonical map*

$$\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^-_{\wp}\mathcal{U} = \mathbf{Spec}_{\wp}\mathcal{U} \bigcap \mathbf{Spec}^-(X), \quad [P] \longmapsto \langle P \rangle,$$

is an isomorphism.

Proof. (a) Let $\mathcal{P}$ be an element of $\mathbf{Spec}_{\wp}\mathcal{U}$, and let $q^* = q^*_{\mathcal{P}}$ be the canonical localization functor $C_X \longrightarrow C_X/\mathcal{P}$. We denote by $J_{\mathcal{P}}$ the set of all $i \in J$ such that $Ker(u_i^*) \subseteq \mathcal{P}$. Set $J^{\mathcal{P}} = J - J_{\mathcal{P}}$. Fix an object P of C_X such that $q^*(P)$ is a quasi-final object of the category $C_X/\mathcal{P}$.

There exists an object L of the subcategory $\bigcap_{i\in J^{\mathcal{P}}} Ker(u_i^*)$ such that $q^*(L)$ is isomorphic to the quasi-final object $q^*(P)$.

By definition, $j \in J^{\mathcal{P}}$ iff $Ker(u_j^*)$ is not contained in $Ker(q^*)$. Let M be an object of $Ker(u_j^*)$ which does not belong to $Ker(q^*)$. Then $q^*(M) \succ q^*(P)$, i.e. there exists a diagram

$$q^*(M^{\oplus I}) \overset{\mathfrak{j}}{\longleftarrow} q^*(K) \overset{\mathfrak{e}}{\longrightarrow} q^*(P) \tag{4}$$

whose left arrow is a monomorphism and the right arrow is an epimorphism. To the diagram (4), there corresponds the diagram

$$M^{\oplus I} \overset{s}{\longrightarrow} M' \overset{\mathfrak{j}'}{\longleftarrow} K \overset{\mathfrak{e}'}{\longrightarrow} P' \overset{t}{\longleftarrow} P \tag{5}$$

such that $q^*(s)$ and $q^*(t)$ are invertible and $\mathfrak{j} = q^*(s)^{-1}q^*(\mathfrak{j}')$, $\mathfrak{e} = q^*(t)^{-1}q^*(\mathfrak{e}')$.

Moreover, we can assume that the arrow $\mathfrak{j}'$ in (5) is a monomorphism (see the part (a) of the argument of 6.2). This implies that the projection $K'' = M^{\oplus I} \times_{M'} K \longrightarrow M^{\oplus I}$ is a monomorphism too. Thus, we obtain the diagram

$$M^{\oplus I} \overset{\mathfrak{j}''}{\longleftarrow} K'' \overset{\mathfrak{e}''}{\longrightarrow} P'_j \overset{s'}{\longrightarrow} P' \overset{t}{\longleftarrow} P \tag{6}$$

in which the arrow $\mathfrak{j}''$ is a monomorphism, the object P'_j is the image of the composition $K'' \longrightarrow K \longrightarrow P'$ and $\mathfrak{e}''$ and s' are resp. an epimorphism and a monomorphism. It follows that $q^*(s')$ is invertible. Let P_j denote the intersection (pull-back) of $P'_j \overset{s'}{\longrightarrow} P'$ and the image of $P \overset{t}{\longrightarrow} P'$. If follows that P_j is a subquotient of $M^{\oplus I}$ (see (6)). Since $M^{\oplus I} \in ObKer(u_j^*)$, and $Ker(u_j^*)$, being a thick subcategory, is closed under taking subquotients, the object P_j belongs to $Ker(u_j^*)$. It follows from the construction that $q^*(P_j)$ is isomorphic to the quasi-final object $q^*(P)$.

Observe that if the initially chosen object P belongs to some thick subcategory $\mathbb{T}$ (or any subcategory $\mathbb{T}$ closed under subquotients taken in C_X), then the object P_j, being a subquotient of P, belongs to the intersection $\mathbb{T} \bigcap Ker(u_j^*)$.

We replace P by P_j and apply the same construction to another $i \in J^{\mathcal{P}}$. As a result, we obtain an object, P_{ij} which belongs to the intersection $Ker(u_j^*) \bigcap Ker(u_i^*)$ and such that $q^*(P_{ij}) \simeq q^*(P_j)$. Continuing this procedure until all elements of $J^{\mathcal{P}}$ are exhausted, we obtain a $Ker(q^*)$-torsion free object $\widetilde{P}$ of the subcategory $\bigcap_{i\in J^{\mathcal{P}}} Ker(u_i^*)$ such that $q^*(\widetilde{P})$ is isomorphic to $q^*(P)$; so that $q^*(\widetilde{P})$ is a quasi-final object of the category $C_X/\mathcal{P}$.

(b) This shows, in particular, that a point $\mathcal{P}$ of $\mathbf{Spec}^1(X)$ can be regarded as a point of $\mathbf{Spec}^1(V_{\mathcal{P}})$, where $V_{\mathcal{P}}$ is defined by $C_{V_{\mathcal{P}}} = \bigcap_{i\in J^{\mathcal{P}}} Ker(u_i^*)$. The 'space' $V_{\mathcal{P}}$ is interpreted as the complement to the 'open subspace' $\bigcup_{i\in J^{\mathcal{P}}} U_i$ in X.

The set of inverse image functors $\{C_X \overset{u_i^*}{\longrightarrow} C_{U_i} \mid i \in J_{\mathcal{P}}\}$ unduces a cover (i.e. a conservative set of localizations) of $V_{\mathcal{P}}$.

In fact, to any morphism u_i, $i \in J_{\mathcal{P}}$, there corresponds a localization of $C_{V_{\mathcal{P}}}$ with the kernel $Ker(u_i^*)\bigcap C_{V_{\mathcal{P}}}$ (which is a thick subcategory of the category $C_{V_{\mathcal{P}}}$). The intersection of all these kernels coincides with

$$\Big(\bigcap_{i\in J_{\mathcal{P}}} Ker(u_i^*)\Big)\bigcap C_{V_{\mathcal{P}}} = \Big(\bigcap_{i\in J_{\mathcal{P}}} Ker(u_i^*)\Big)\bigcap\Big(\bigcap_{j\in J^{\mathcal{P}}} Ker(u_j^*)\Big) = \bigcap_{i\in J} Ker(u_i^*)$$

and $\bigcap_{i\in J} Ker(u_i^*) = 0$, because the set of functors $\{u_i^* \mid i \in J\}$ is conservative by hypothesis.

(c) Replacing X by $V_{\mathcal{P}}$, we shall assume for the rest of the argument that $J_{\mathcal{P}} = J$, i.e. $Ker(u_i^*) \subseteq \mathcal{P}$ for any $i \in J$.

Let L be an object of C_X such that $q^*(L)$ is a quasi-final object of the local category $C_X/\mathcal{P}$. There exists an object P of the category C_X such that for every $i \in J$, the object $u_i^*(P)$ belongs to the spectrum, $Spec(U_i)$, of U_i and such that $q^*(P)$ is isomorphic to $q^*(L)$. In particular, $q^*(P)$ is a quasi-final object of the local category $C_X/\mathcal{P}$.

(c1) Fix an $i \in J$. By hypothesis, $Ker(u_i^*) \subseteq \mathcal{P}$ and $\mathcal{P}/Ker(u_i^*)$ belongs to $\mathbf{Spec}(U_i)$, i.e. there exists an object P_i of the category C_X such that $\mathcal{P}/Ker(u_i^*) = \langle u_i^*(P_i)\rangle$. The object $q^*(P_i)$ is quasi-final, hence equivalent to the object $q^*(L)$. In particular, $q^*(P_i) \succ q^*(L)$. The latter relation is expressed by the following diagram:

$$P_i^{\oplus n} \xleftarrow{\ \mathfrak{j}\ } K \xrightarrow{\ \mathfrak{e}\ } P_i' \xrightarrow{\ \mathfrak{j}'\ } \widetilde{P}' \xleftarrow{\ \mathfrak{e}'\ } L \qquad (7)$$

in which $\mathfrak{j}$, $\mathfrak{j}'$ are monomorhisms, $\mathfrak{e}$, $\mathfrak{e}'$ are epimorphisms, and the arrows $\mathfrak{j}'$, $\mathfrak{e}'$ belong to $\Sigma_{q^*} = \{s \in HomC_X \mid q^*(s) \text{ is invertible}\}$.

We claim that $u_i^*(P_i)$ is equivalent (with respect to $\succ$) to $u_i^*(P_i')$.

Let q_i^* denote the localization functor $C_{U_i} \longrightarrow C_X/\mathcal{P} = C_{U_i}/(\mathcal{P}/Ker(u_i^*))$; so that $q^* \simeq q_i^* u_i^*$. If $u_i^*(P_i') \not\succ u_i^*(P_i)$, then $q^*(P_i') \simeq q_i^*(u_i^*(P_i')) = 0$ which contradicts to the isomorphism (following from (7)) $q^*(P_i') \simeq q^*(L)$ and the fact that, by definition, $q^*(L)$ is a nonzero object. Thus, $u_i^*(P_i') \succ u_i^*(P_i)$. The inverse relation, $u_i^*(P_i) \succ u_i^*(P_i')$, follows from the relation $P_i \succ P_i'$ (see (7)) and the fact that any exact functor, in particular u_i^*, preserves the relation $\succ$.

Notice that since the arrows $\mathfrak{j}'$, $\mathfrak{e}'$ of the diagram (7) belong to Σ_{q^*}, the object $q^*(P_i')$ is isomorphic to $q^*(L)$.

(c2) Now fix another element, j, of J. Repeating the argument of (c1) with the object L replaced by the object P_i', we obtain an object P_j' of C_X such that $P_j \succ P_j'$ and there exists a diagram

$$P_i' \xrightarrow{\ \mathfrak{e}\ } M \xleftarrow{\ \mathfrak{j}\ } P_j' \qquad (8)$$

in which $\mathfrak{e}$ is an epimorphism, $\mathfrak{j}$ is a monomorphism, and both arrows belong to Σ_{q^*} (i.e. q^* maps them to isomorphisms). By the argument (c1), the object $u_j^*(P_j')$ is equivalent to $u_j^*(P_j)$, and $q^*(P_j'$ is isomorphic to $q^*(P_j')$. Since $P_i' \succ P_j'$ (see (8) above) and u_i^* is an exact functor, $u_i^*(P_i') \succ u_i^*(P_j')$.

On the other hand, $u_i^*(P_j') \succ u_i^*(P_i')$, because if not, $u_i^*(P_j')$ would belong to $Ker(q_i^*) = \langle u_i^*(P_i')\rangle$. But, $q_i^* u_i^*(P_j') \simeq q^*(P_j') \simeq q^*(P_i') \simeq q^*(L) \neq 0$..

(c3) Replacing P'_i by P'_j, we repeat the procedure, and continue doing so until J will be exhausted. As a result, we obtain an object P of C_X such that $q^*(P) \simeq q^*(L)$, and $u_i^*(P)$ belongs to $Spec(U_i)$ for every $i \in J$.

(d) Since X has the property (sup), every Serre subcategory of C_X, in particular $Ker(q^*)$, is coreflective (cf. 2.5.5). Replacing the object P constructed in (c) by its quotient by $Ker(q^*)$-torsion of P, we assume that P is $Ker(q^*)$-torsion free. Since $Ker(u_i^*)$ is contained in $Ker(q^*)$ for all $i \in J$ (cf (c) above), the object P is $Ker(u_i^*)$-torsion free for all $i \in J$. It follows from 6.2.1 that P belongs to $Spec(X)$. ■

6.3.1. Note. The hypothesis that X has the property (sup) was used only in the last step of the proof of 6.3 in order to ensure the existence of a $Ker(q^*)$-torsion free object P such that $q^*(P)$ is a quasi-final object. Thus, the assertion 6.3 is valid without any hypothesis on X (except for C_X being abelian), if we will consider only $\mathcal{P} \in \mathbf{Spec}_\wp \mathcal{U}$ and $[M] \in \mathbf{Spec}(X)$ such that the subcategories $\mathcal{P}$ and $\langle M \rangle$ are coreflective.

6.4. The combinatorial spectrum of a locally quasi-compact 'space'. Fix a quasi-presite $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a subcategory of $|\mathfrak{Ab}|^o$ and τ is a quasi-pretopology on $\mathcal{A}$ such that for any cover $\{U_j \xrightarrow{u_j} X \mid j \in J\}$, the family of inverse image functors $\{C_X \xrightarrow{u_j^*} C_{U_j} \mid j \in J\}$ is a conservative set of exact localizations.

Recall that for any object X of $\mathcal{A}$, we denote by $\Sigma^\tau X$ the set of all arrows $U \longrightarrow X$ which belong to some τ-cover of X. And $\Sigma_\tau X$ denotes the class of all morphisms $Y \xrightarrow{f} X$ such that the pull-back of a τ-cover along f is a τ-cover.

6.4.1. Proposition. *Let X be an object of $\mathcal{A}$ with the property (sup).*

(a) If X is τ-quasi-compact, then the canonical map $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^-_\wp(X, \tau_X)$ is an isomorphism.

(b) Suppose X has the property $\Sigma^\tau X \subseteq \Sigma_\tau X$. Then $\mathbf{Spec}^-_\wp(X, \tau_X) = \mathbf{Spec}^-_\wp \mathcal{U}$ for any τ-cover $\mathcal{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ such that all U_i are quasi-compact.

Proof. Let $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ be τ-covers of X. Suppose $\mathcal{U}$ is a subcover of $\widetilde{\mathcal{U}}$. Then there are inclusions

$$\mathbf{Spec}(X) \hookrightarrow \mathbf{Spec}^-_\wp \widetilde{\mathcal{U}} \hookrightarrow \mathbf{Spec}^-_\wp \mathcal{U}. \tag{1}$$

(a) Suppose X is τ-quasi-compact, that is any τ-cover $\widetilde{\mathcal{U}}$ of X has a finite subcover, say $\mathcal{U}$. By 6.3, the embedding $\mathbf{Spec}(X) \hookrightarrow \mathbf{Spec}^-_\wp \mathcal{U}$ (which is the composition of the maps of (1)) is an isomorphism. Therefore $\mathbf{Spec}(X) \hookrightarrow \mathbf{Spec}^-_\wp \widetilde{\mathcal{U}}$ is an isomorphism. This implies that the map $\mathbf{Spec}(X) \hookrightarrow \mathbf{Spec}^-_\wp(X, \tau_X) := \operatorname{colim}\big(\mathbf{Spec}^-_\wp \widetilde{\mathcal{U}} \mid \widetilde{\mathcal{U}} \in Cov_\tau X\big)$ is an isomorphism.

(b) Fix a τ-cover $\mathcal{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ such that all U_i are quasi-compact. Let $\mathcal{V} = \{V_\nu \xrightarrow{\mathfrak{v}_i} X \mid \nu \in \mathfrak{J}\}$ be any other τ-cover of X. The inclusion $\Sigma^\tau X \subseteq \Sigma_\tau X$ garantees that the covers $\mathcal{V}$ and $\mathcal{U}$ have a common refinement, $\mathcal{V} \times_X \mathcal{U} = \{U_{i\nu} \to X \mid (i, \nu) \in J \times \mathfrak{J}\}$. Here $U_{i\nu}$ denotes $U_i \times_X V_\nu$. By a general standard argument, the canonical morphism

$$\operatorname{colim}\big(\mathbf{Spec}_\wp(\mathcal{V} \times_X \mathcal{U}) \mid \mathcal{V} \in Cov_\tau X\big) \longrightarrow \operatorname{colim}\big(\mathbf{Spec}_\wp(\mathcal{V}) \mid \mathcal{V} \in Cov_\tau X\big)$$

is an isomorphism, and, by definition, $\operatorname{colim}\big(\mathbf{Spec}_\wp(\mathcal{V}) \mid \mathcal{V} \in Cov_\tau X\big) = \mathbf{Spec}_\wp(X, \tau_X)$.

On the other hand, for any τ-cover $\mathcal{V} = \{V_\nu \xrightarrow{\mathfrak{v}_i} X \mid \nu \in \mathfrak{J}\}$ of X, the τ-cover $\mathcal{V} \times_X \mathcal{U} \longrightarrow X$ is a refinement of the τ-cover $\mathcal{U}$. Thus, we have a canonical morphism $\mathbf{Spec}^-_\wp \mathcal{U} \longrightarrow \mathbf{Spec}^-_\wp(\mathcal{V} \times_X \mathcal{U})$. This morphism is an isomorphism.

In fact, for any $i \in J$, the set of morphisms $\{U_{i\nu} \to U_i \mid \nu \in \mathfrak{J}\}$ is a cover of U_i. Since U_i is, by hypothesis, quasi-compact, this cover has a finite subcover, $\widetilde{\mathcal{U}}_i$. There are obvious inclusions (see (1) above):

$$\mathbf{Spec}(U_i) \hookrightarrow \mathbf{Spec}^-_\wp(\mathcal{V} \times_X \mathcal{U}) \hookrightarrow \mathbf{Spec}^-_\wp(\widetilde{\mathcal{U}}_i). \tag{2}$$

Since the τ-cover $\widetilde{\mathcal{U}}_i$ is finite, the canonical map $\mathbf{Spec}(U_i) \hookrightarrow \mathbf{Spec}^-_\wp(\widetilde{\mathcal{U}}_i)$, the composition of the arrows (2), is an isomorphism. Therefore the map

$$\mathbf{Spec}(U_i) \hookrightarrow \mathbf{Spec}^-_\wp(\mathcal{V} \times_X \mathcal{U}) \tag{3}$$

in (2) is an isomorphism. Since all arrows here are functorial, the isomorphisms (3), $i \in J$, induce an isomorphism $\mathbf{Spec}^-_\wp \mathcal{U} \longrightarrow \mathbf{Spec}^-_\wp(\mathcal{V} \times_X \mathcal{U})$, Therefore, the canonical morphism

$$\mathbf{Spec}^-_\wp \mathcal{U} \longrightarrow \operatorname{colim}\big(\mathbf{Spec}^-_\wp(\mathcal{V} \times_X \mathcal{U}) \mid \mathcal{V} \in Cov_\tau X\big) \tag{4}$$

is an isomorphism. This shows that $\mathbf{Spec}^-_\wp \mathcal{U} \longrightarrow \mathbf{Spec}^-_\wp(X, \tau_X)$ is an isomorphism. ■

6.5. Extended spectra.

6.5.1. The extended complete spectrum and the extended S-spectrum. For any $X \in Ob|\mathfrak{Ab}|^o$, we denote by $\mathbf{Spec}^1_\star X$ the ordered set $\mathbf{Spec}^1(X) \cup \{\star_X\}$, where $\mathbf{Spec}^1(X)$ is a complete spectrum of the 'space' X and $\star_X$ is the added point.

If $\mathbf{Spec}^1(X)$ is realized as the set of 'geometric points', i.e. morphisms $\mathbf{x} \xrightarrow{q} X$ from local 'spaces' such that q^* is an exact localization, then $\star_X$ is becomes the (unique) morphism from the zero 'space' represented by the zero category. Notice that the zero 'space' is both final and initial object of the category $|\mathfrak{Ab}|^o$.

If $\mathbf{Spec}^1(X)$ is realized as the set thick subcategories, $\mathbf{P}$, of C_X such that $C_X/\mathbf{P}$ is a local category, then the zero 'space' corresponds to the unproper thick subcategory $C_X \xrightarrow{Id} C_X$; i.e. $\star_X$ is realized as the category C_X itself.

We define the *extended S-spectrum*, by $\mathbf{Spec}^-_\star(X) = \mathbf{Spec}^-(X) \cup \{\star_X\}$.

6.5.2. The extended spectrum. If $\mathfrak{Y}$ is a (preordered) subset of $\mathbf{Spec}^-(X)$, the symbol $\mathfrak{Y}_\star$ will denote $\mathfrak{Y} \cup \{\star_X\}$. In particular, we have the extended spectrum, $\mathbf{Spec}_\star(X) = \mathbf{Spec}(X) \cup \{\star_X\}$.

The extended spectrum $\mathbf{Spec}_\star(X)$ can be naturally obtained via an extended version of the spectrum $Spec(X)$ (cf. 2.2). Namely, we define $Spec_\star(X)$ by adjoining to $Spec(X)$ the zero object and imposing its minimality. Notice that $\langle 0 \rangle$ is the empty category, $\emptyset$. So that passing from $Spec_\star(X)$ to $\mathbf{Spec}_\star(X)$ via the map $P \longmapsto \langle P \rangle$ gives another realization of the initial element, dual to the one above: instead of the trivial subcategory $C_X \xrightarrow{Id} C_X$, we use the trivial subcategory $\emptyset \hookrightarrow C_X$.

6.5.3. Functorial properties of the extended spectra. Let $|\mathfrak{L}_\mathfrak{c}\mathfrak{Ab}|^o$ denote the subcategory of the category $|\mathfrak{Ab}|^o$ formed by all morphisms $X \longrightarrow Y$ of $|\mathfrak{Ab}|^o$ whose inverse image functors are exact localizations (in particular, $Ob|\mathfrak{L}_{\mathfrak{e}}\mathfrak{Ab}|^o = Ob|\mathfrak{Ab}|^o$). On the other hand, let $\mathfrak{Ord}_\star$ denote the category whose objects are preordered sets with the initial (i.e. the smallest) element and arrows are morphisms of ordered sets which map the initial element to the initial element. Finally, let $\mathfrak{Ord}_\star$ denote the category objects of which are preorders (i.e. categories with no more than one arrow between any two objects) with an initial object and morphisms are functors which map initial objects to initial objects.

6.5.3.1. Proposition. *(a) The map* $X \longmapsto \mathbf{Spec}^-_\star(X)$ *extends to a contravariant functor,* $\mathbf{Spec}^-_\star$*, from the category* $|\mathfrak{L}_{\mathfrak{e}}\mathfrak{Ab}|^o$ *to the category* $\mathfrak{Ord}_\star$ *of preordered sets with the smallest element.*

(b) The map $X \longmapsto Spec_\star(X)$ *extends to a contravariant pseudo-functor from the category* $|\mathfrak{L}_{\mathfrak{e}}\mathfrak{Ab}|^o$ *to the category* $\mathfrak{O}_\star Cat$ *of preorders with an initial object. This functor gives a rise to a subfunctor,* $\mathbf{Spec}_\star$*, or the functor* $\mathbf{Spec}^-_\star$*.*

Proof. (a) Let $Y \xrightarrow{f} X$ be a morphism of $|\mathfrak{L}_{\mathfrak{e}}\mathfrak{Ab}|^o$; i.e. $C_X \xrightarrow{f^*} C_Y$ is an exact localization. We define a map $f^- : \mathbf{Spec}^-_\star X \longrightarrow \mathbf{Spec}^-_\star(Y)$ as follows. Let $\mathbf{P} \in \mathbf{Spec}^-(X)$. If $Ker(f^*) \subseteq \mathbf{P}$, then $f^-(\mathbf{P}) = (f^*(\mathbf{P})^-$ – the minimal Serre subcategory spanned by the image of $\mathbf{P}$. If C_Y is identified with the quotient category $C_X/Ker(f^*)$, then $f^-(\mathbf{P})$ is identified with the subcategory $\mathbf{P}/Ker(f^*)$. By transitivity of localizations, the quotient category $(C_X/Ker(f^*))/(\mathbf{P}/Ker(f^*))$ is equivalent to $C_X/\mathbf{P}$, hence it is local. This shows that, $f^-(\mathbf{P})$ belongs, indeed, to the subset $\mathbf{Spec}^-(Y)$ of $\mathbf{Spec}^-_\star(Y)$. If $Ker(f^*)$ is not contained in $\mathbf{P}$, then f^- maps the point $\mathbf{P}$ to the initial object $\star_Y$. It is required that $f^-(\star_X) = \star_Y$. It is easy to check that thus defined map $f \longmapsto f^-$ is a functor $\left(|\mathfrak{L}_{\mathfrak{e}}\mathfrak{Ab}|^o\right)^{op} \longrightarrow \mathfrak{Ord}_\star$.

(b) For any morphism $Y \xrightarrow{f} X$ of the category $|\mathfrak{L}_{\mathfrak{e}}\mathfrak{Ab}|^o$ we choose an inverse image functor, f^*. The functor f^* induces a morphism (a functor) $Spec_\star(X) \xrightarrow{\widetilde{f}^a} Spec_\star(Y)$ between preorders (see the argument of 2.4.2) which, obviously, maps initial objects to initial objects: $f^*(0) = 0$. Being exact, the functor f^* respects the preorder $\succ$, hence defines, via $P \longmapsto \langle P \rangle$, a map $\mathbf{Spec}(X) \cup \emptyset \xrightarrow{f^a} \mathbf{Spec}(Y) \cup \emptyset$ uniquely determined by the commutativity of the diagram

$$\begin{array}{ccc} Spec_\star(X) & \xrightarrow{\widetilde{f}^a} & Spec_\star(Y) \\ \big\downarrow & & \big\downarrow \\ \mathbf{Spec}(X) \cup \emptyset & \xrightarrow{f^a} & \mathbf{Spec}(Y) \cup \emptyset \end{array}$$

It remains to identify the $\emptyset$ with $\star_X$. ■

6.5.4. The extended upper spectrum and combinatorial spectrum of a quasi-topological 'space'. If X is an object of quasi-presite $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a subcategory of $|\mathfrak{Ab}|^o$ and τ-covers are conservative families of localizations, then we have the extended upper spectrum, $\mathbf{Spec}_\star(X, \tau_X) = \mathbf{Spec}(X, \tau_X) \cup \{\star_X\}$, and the extended combinatorial spectrum, $\mathbf{Spec}_{\wp\star}(X, \tau_X) = \mathbf{Spec}_\wp(X, \tau_X) \cup \{\star_X\}$ of the quasi-topological 'space' (X, τ_X). We define the extended versions of the upper S-spectrum and the lower (combinatorial) S-spectrum as intersections of the respective spectra with the family of all Serre subcategories of C_X. The inclusions

$$\begin{array}{ccccccc}
\mathbf{Spec}_\star(X) & \longrightarrow & \mathbf{Spec}^-_{\wp\star}(X,\tau_X) & \longrightarrow & \mathbf{Spec}^-_\star(X,\tau_X) & \longrightarrow & \mathbf{Spec}^-_\star(X) \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
 & & \mathbf{Spec}_{\wp\star}(X,\tau_X) & \longrightarrow & \mathbf{Spec}_\star(X,\tau_X) & \longrightarrow & \mathbf{Spec}^1_\star(X)
\end{array} \qquad (1)$$

follow from the definitions of these spectra.

Let $\mathfrak{L}_\mathfrak{e}\mathcal{A}$ denote the subcategory of the category $\mathcal{A}$ formed by all morphisms $X \xrightarrow{f} Y$ whose inverse image is an exact localization. Since τ-covers consist of exact localizations, quasi-topological 'spaces' on $(\mathfrak{L}_\mathfrak{e}\mathcal{A},\tau)$ and on $(\mathcal{A},\tau)$ are same.

6.5.4.1. Proposition. *The maps*

$$(X,\tau_X) \longmapsto \mathbf{Spec}_\star(X,\tau_X) \quad \textit{and} \quad (X,\tau_X) \longmapsto \mathbf{Spec}_{\wp\star}(X,\tau_X)$$

are extended to contravariant functors from the category $\mathfrak{Top}_{(\mathfrak{L}_\mathfrak{e}\mathcal{A},\tau)}$ of quasi-topological 'spaces' on $(\mathfrak{L}_\mathfrak{e}\mathcal{A},\tau)$ to the category $\mathfrak{Ord}_\star$ of preordered sets with the smallest element.

Proof. The fact can be deduced from 6.5.3.1. Details are left to the reader. ■

6.5.5. Towards computations of the (extended) combinatorial spectrum. Let $\mathcal{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a cover of a 'space' X. For every subset I of J, let C_{U_I} be the quotient of the category C_X by the thick subcategory $\mathbb{T}_I = \bigcap_{i\in I} Ker(u_i^*)$. The localization functor $C_X \xrightarrow{u_I^*} C_{U_I}$ is regarded as an inverse image functor of a morphism $U_I \xrightarrow{u_I} X$. If $\mathfrak{I}$ is another subset of J, then we have a diagram

$$\begin{array}{ccccc}
 & & U_\mathfrak{I} & & \\
 & & \big\downarrow & & \\
U_I & \longrightarrow & U_{I\cap\mathfrak{I}} & \longrightarrow & X
\end{array} \qquad (1)$$

Let $J_\mathfrak{ft}$ denote the preordered (with respect to the inclusion) set of finite subsets of J. The map $\mathfrak{I} \longmapsto U_\mathfrak{I}$ is a functor from $J_\mathfrak{ft}$ to the category of 'subspaces' of the 'space' X. We denote by $\mathcal{U}^\mathfrak{ft}$ the set of morphisms $\{U_\mathfrak{I} \xrightarrow{u_\mathfrak{I}} X \mid \mathfrak{I} \in J_\mathfrak{ft}\}$.

6.5.5.1. Lemma. *There is a natural isomorphism* $\mathbf{Spec}^-_\wp(\mathcal{U}^\mathfrak{ft}) \simeq \mathbf{Spec}^-_\wp\mathcal{U}$.

Proof. For any $\mathfrak{I} \in J_\mathfrak{ft}$, the set of morphisms $\{U_i \xrightarrow{u_i} X \mid i \in \mathfrak{I}\}$ induces a finite conservative set of localizations $\mathcal{U}^\mathfrak{I} = \{U_i \to U_\mathfrak{I} \mid i \in \mathfrak{I}\}$. By 6.3, the natural map $\mathbf{Spec}(U_\mathfrak{I}) \longrightarrow \mathbf{Spec}^-_\wp(\mathcal{U}^\mathfrak{I})$ is an isomorphism. This implies that a canonical map $\mathbf{Spec}^-_\wp(\mathcal{U}^\mathfrak{ft}) \longrightarrow \mathbf{Spec}^-_\wp\mathcal{U}$ (which is due to the fact that the cover $\mathcal{U}$ is a refinement of the cover $\mathcal{U}^\mathfrak{ft}$) is an isomorphism. Details are left to the reader. ■

Let $\mathbf{Spec}_{\wp\star}\mathcal{U}$ denote the extended spectrum of the cover $\mathcal{U}$ defined by $\mathbf{Spec}_{\wp\star}\mathcal{U} = \mathbf{Spec}_\wp\mathcal{U} \cup \{\star_X\}$. In the notations above, we have the following

6.5.5.2. Lemma. *Let $\mathcal{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a τ-cover of a 'space' X, and let $\mathcal{U}^{\mathfrak{ft}} = \{U_{\mathfrak{J}} \xrightarrow{u_{\mathfrak{J}}} X \mid \mathfrak{J} \in J_{\mathfrak{ft}}\}$ be the associated with $\mathcal{U}$ filtered cover. There is a natural isomorphism*

$$\lim_{\mathfrak{J} \in J_{\mathfrak{ft}}} \mathbf{Spec}^-_{\wp\star} U_{\mathfrak{J}} \xrightarrow{\sim} \mathbf{Spec}^-_{\wp\star} \mathcal{U}. \tag{2}$$

Proof. Thanks to 6.5.5.1, it suffices to show that the canonical morphism

$$\lim_{\mathfrak{J} \in J_{\mathfrak{ft}}} \mathbf{Spec}^-_{\wp\star} U_{\mathfrak{J}} \xrightarrow{\phi_{\mathcal{U}}} \mathbf{Spec}^-_{\wp\star} \mathcal{U}^{\mathfrak{tt}} \tag{3}$$

is an isomorphism. But, this follows from the definition of the morphism (3) which assigns to every element $(\mathbf{P}_{\mathfrak{J}} \mid \mathfrak{J} \in J_{\mathfrak{ft}})$ of $\lim\limits_{\mathfrak{J} \in J_{\mathfrak{ft}}} \mathbf{Spec}^-_{\wp\star} U_{\mathfrak{J}}$ the element $\mathbf{P}$ of $\mathbf{Spec}^-_{\star}(X)$ uniquely defined by the condition that the image of $\mathbf{P}$ in $\mathbf{Spec}^-_{\star} U_{\mathfrak{J}}$ coincides with $\mathbf{P}_{\mathfrak{J}} \in \mathbf{Spec}^-_{\wp\star} U_{\mathfrak{J}}$. The map which assigns to each element $\mathbf{P}$ of $\mathbf{Spec}^-_{\wp\star} \mathcal{U}^{\mathfrak{tt}}$ the element $(\mathbf{P}_{\mathfrak{J}} \mid \mathfrak{J} \in J_{\mathfrak{ft}})$ of $\prod\limits_{\mathfrak{J} \in J_{\mathfrak{ft}}} \mathbf{Spec}^-_{\star} U_{\mathfrak{J}}$, where $\mathbf{P}_{\mathfrak{J}}$ is the image of $\mathbf{P}$ in $\mathbf{Spec}^-_{\star} U_{\mathfrak{J}}$, takes values in $\lim\limits_{\mathfrak{J} \in J_{\mathfrak{ft}}} \mathbf{Spec}^-_{\wp\star} U_{\mathfrak{J}}$ and (its corestriction to $\lim\limits_{\mathfrak{J} \in J_{\mathfrak{ft}}} \mathbf{Spec}^-_{\star} U_{\mathfrak{J}}$) is the inverse map to (3). ∎

6.5.5.3. Proposition. *Let $(\mathcal{A}, \tau)$ be a quasi-presite such that $\mathcal{A}$ is a subcategory of $|\mathfrak{Ab}|^o$ and τ-covers are conservative sets of exact localizations. Let $\mathcal{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a τ-cover of a 'space' X, and let $\mathcal{U}^{\mathfrak{t}} = \{U_{\mathfrak{J}} \xrightarrow{u_{\mathfrak{J}}} X \mid \mathfrak{J} \in J_{\mathfrak{ft}}\}$ be the associated with $\mathcal{U}$ filtered cover. Suppose X has the property (sup) and $\Sigma_\tau X \subseteq \Sigma^\tau X$. If U_i is quasi-compact for all $i \in J$, then there is a natural isomorphism*

$$\lim_{\mathfrak{J} \in J_{\mathfrak{ft}}} \mathbf{Spec}^-_{\wp\star} U_{\mathfrak{J}} \xrightarrow{\sim} \mathbf{Spec}^-_{\wp\star}(X, \tau_X). \tag{4}$$

Proof. The assertion follows from 6.5.5.2 and 6.4.1. ∎

7. Closed coimmersions and immersions. Zariski pretopology.

7.1. Closed coimmersions. We call a morphism $X \xrightarrow{\mathfrak{v}} V_{\mathfrak{v}}$ a *closed coimmersion* if its inverse image functor (is fully faithful and) induces an equivalence between $C_{V_{\mathfrak{v}}}$ and a coreflective topologizing subcategory of the category C_X. In particular, $\mathfrak{v}$ has a direct image functor (a right adjoint to $\mathfrak{v}^*$).

Notice that the spectrum is (contravariantly) functorial with respect to closed coimmersions: to any closed coimmersion $X \longrightarrow Y$, there corresponds a natural embedding $\mathbf{Spec}(Y) \hookrightarrow \mathbf{Spec}(X)$ and this correspondence commutes with the composition.

This functoriality with respect to closed coimmersions extends to the S-spectrum.

7.2. Quasi-pretopologies associated with families of closed coimmersions. Let $C_{V^-_{\mathfrak{v}}}$ be the Serre subcategory of C_X generated by the image of $C_{V_{\mathfrak{v}}}$. Let $C_{U_{\mathfrak{v}}}$ be the quotient $C_X/C_{V^-_{\mathfrak{v}}}$ and $u^*_{\mathfrak{v}}$ the localization functor. We have an exact sequence of categories and functors $C_{V_{\mathfrak{v}}^-} \longrightarrow C_X \longrightarrow C_{U_{\mathfrak{v}}}$ which corresponds to the first two arrows of the diagram of 'spaces'

$$U_{\mathfrak{v}} \xrightarrow{u_{\mathfrak{v}}} X \xrightarrow{\mathfrak{v}^-} V_{\mathfrak{v}}^- \longrightarrow V_{\mathfrak{v}}. \quad (1)$$

Let $\mathcal{A}$ be a subcategory of $|\mathfrak{Ab}|^o$. For any $X \in Ob\mathcal{A}$, we choose a family $\mathfrak{C}_X$ of closed coimmersions. For any element $X \xrightarrow{\mathfrak{v}} V_{\mathfrak{v}}$ of $\mathfrak{C}_X$, we declare the morphism $U_{\mathfrak{v}} \xrightarrow{u_{\mathfrak{v}}} X$ being an element of a cover, an 'open subspace'. Covers are sets $\{U_{\mathfrak{v}} \xrightarrow{u_{\mathfrak{v}}} X \mid \mathfrak{v} \in \mathfrak{V}\}$ of 'open subspaces' such that the family of inverse image functors $\{u_{\mathfrak{v}}^* \mid \mathfrak{v} \in \mathfrak{V}\}$ is conservative.

We assume that $\mathfrak{C}_X$ contains the zero coimmersion $X \longrightarrow \bullet$. Here $C_{\bullet} = 0$. The corresponding open 'subspace' is $X \xrightarrow{id_X} X$.

7.2.1. A special case. If $\mathcal{A} = |\mathfrak{Ab}|^o$ and for any 'space' X the family $\mathfrak{C}_X$ consists of morphisms corresponding to inclusions of all topologizing subcategories of C_X, then the resulting quasi-pretopology is (the restriction to $\mathcal{A}$ of) the quasi-pretopology $\tau^{\mathfrak{L}}$ of exact localizations defined in 1.5.2.

Notice that $\mathfrak{C}_X$ is closed with respect to all intersections (which are colimits of 'small' families $\{X \xrightarrow{\mathfrak{v}} V_{\mathfrak{v}} \mid \mathfrak{v} \in \mathfrak{V}\}$ of arrows of $\mathfrak{C}_X$) and with respect to the Gabriel multiplication (cf. 4.2). It follows from 4.1 that the 'open subspace' corresponding to the intersection of a finite family $\{X \xrightarrow{\mathfrak{v}} V_{\mathfrak{v}} \mid \mathfrak{v} \in \mathfrak{V}\}$ of arrows of $\mathfrak{C}_X$ is the union (i.e. colimit) of the corresponding family $\{U_{\mathfrak{v}} \xrightarrow{u_{\mathfrak{v}}} X \mid \mathfrak{v} \in \mathfrak{V}\}$ of 'open subspaces'.

The 'open subspace' corresponding to the Gabriel product of two closed coimmersions, $V_{\mathfrak{v}} \xleftarrow{\mathfrak{v}} X \xrightarrow{\mathfrak{w}} V_{\mathfrak{w}}$, is the intersection of the associated 'open subspaces', i.e. the fibered product of $U_{\mathfrak{v}} \xrightarrow{u_{\mathfrak{v}}} X \xleftarrow{u_{\mathfrak{w}}} U_{\mathfrak{w}}$ (see the paragraph preceeding to 4.3).

7.2.2. The quasi-pretopology $\tau_{\mathfrak{z}}$. Let the family of closed coimmersions $\mathfrak{C}_X$ consists of morphisms $X \longrightarrow |[M]|$, where M is an object of C_X of finite type, $[M]$ the minimal topologizing coreflective subcategory of C_X containing M, and $C_{|[M]|} = [M]$.

If $C_X = R{-}mod$ for a commutative ring R, then $\mathbf{Spec}(X) = SpecR$, and the topology on $SpecR$ corresponding to $\tau_{\mathfrak{z}}$ coincides with the Zariski topology.

7.2.3. The quasi-pretopology $\tau_{\mathfrak{r}}$. This time, $\mathfrak{C}_X$ consists of the zero morphism and all morphisms $X \longrightarrow |[P]|$, with $P \in Spec(X)$.

If the category C_X has enough objects of finite type (that is every nonzero object of C_X has a nonzero subobject of finite type), then the quasi-pretopology $\tau_{\mathfrak{r}}$ is coarser than the quasi-pretopology $\tau_{\mathfrak{z}}$.

7.3. Closed immersions. Zariski quasi-presite. We call a morphism $V \xrightarrow{\mathfrak{f}} X$ a closed immersion if it is continuous and its direct image functor $\mathfrak{f}_* : C_V \longrightarrow C_X$ identifies the category C_V with a topologizing subcategory of the category C_X. In other words, $\mathfrak{f}_*$ is an inverse image functor of a closed coimmersion.

7.3.1. Proposition. *Let C_X be the category $R-mod$ of R-modules over an associative unital ring R. Then there is a bijective correspondence between isomorphism classes of closed immersions and two-sided ideals of the ring R.*

Proof. To any two-sided ideal α in the ring R, we assign the category $R/\alpha - mod$ of R/α-modules which we identify with the full subcategory of the category $R - mod$

whose objects are modules annihilated by α. It is shown in [R, Ch.3, 6.4.1] that this map establishes the desired bijective correspondence. ■

7.3.2. Zariski quasi-presite. Let $\tau_X^{\mathfrak{Z}}$ be the family of all isomorphism classes of closed immersions to X. We denote the corresponding class of covers of X by $\tau_X^{\mathfrak{Z}}$. This defines the *Zariski* quasi-pretopology on $|\mathfrak{Ab}|^o$.

7.4. Proposition. *Let C_X be the category of quasi-coherent sheaves on a scheme $\mathbf{X} = (\mathcal{X}, \mathcal{O})$. Let τ_X be the Zariski pretopology $\tau^{\mathfrak{Z}}$, and τ^X topology induced by $\tau_X^{\mathfrak{Z}}$ on the spectrum $\mathbf{Spec}(X, \tau_X^{\mathfrak{Z}})$. Then the underlying space $\mathcal{X}$ of the scheme $\mathbf{X}$ is isomorphic to the upper spectrum $(\mathbf{Spec}(X, \tau_X^{\mathfrak{Z}}), \tau^X)$ of the quasi-topological 'space' $(X, \tau_X^{\mathfrak{Z}})$.*

The upper spectrum $\mathbf{Spec}(X, \tau_X^{\mathfrak{Z}})$ coincides with the lower spectrum $\mathbf{Spec}_{\wp}(X, \tau_X^{\mathfrak{Z}})$.

Proof. Let $\mathbf{Y} \xrightarrow{\mathfrak{j}} \mathbf{X}$ be a closed subscheme of the scheme $\mathbf{X}$ with the defining ideal $\mathcal{J}_{\mathbf{Y}}$. Then a direct image functor, $\mathfrak{j}_*$, of the embedding $\mathfrak{j}$ induces an equivalence of the category $Qcoh_{\mathbf{Y}}$ of quasi-coherent sheaves on $\mathbf{Y}$ to the full subcategory $\mathbb{T}_{\mathbf{Y}}$ of the category $C_X = Qcoh_{\mathbf{X}}$ formed by quasi-coherent $\mathcal{O}$-modules annihilated by the ideal $\mathcal{J}_{\mathbf{Y}}$. The functor $\mathcal{M} \longmapsto \mathcal{O}/\mathcal{J}_{\mathbf{Y}} \otimes_{\mathcal{O}} \mathcal{M}$ is a left adjoint to the inclusion functor $\mathbb{T}_{\mathbf{Y}} \hookrightarrow C_X$; i.e. $\mathfrak{j}_*$ is a direct image of a closed immersion in the sense of 7.3.

Conversely, let $\mathbb{T}$ be a subcategory of C_X such that the inclusion functor, $\mathbb{T} \overset{\mathfrak{j}_*}{\hookrightarrow} C_X$, is a direct image functor of a closed immersion (in the sense of 7.3). Then an adjunction morphism $Id_{C_X} \xrightarrow{\eta_{\mathfrak{j}}} \mathfrak{j}_*\mathfrak{j}^*$ is an epimorphism. In particular, we have an epimorphism $\mathcal{O} \longrightarrow \mathfrak{j}_*\mathfrak{j}^*(\mathcal{O})$ of quasi-coherent sheaves with the kernel $\mathcal{J}_{\mathbb{T}}$. Objects of the subcategory $\mathbb{T}$ are precisely those quasi-coherent $\mathcal{O}$-modules which are annihilated by the ideal $\mathcal{J}_{\mathbb{T}}$. Thus, the pretopology on X defined by τ is the same as the pretopology defined by Zariski topology. Therefore the map of sets $\mathcal{X} \longrightarrow \mathbf{Spec}(X, \tau_X)$ is a bijection, which implies that the map $\mathcal{X} \longrightarrow (\mathbf{Spec}(X, \tau_X), \tau^X)$ of topological spaces is an isomorphism.

The equality $\mathbf{Spec}_{\wp}(X, \tau_X^{\mathfrak{Z}}) = \mathbf{Spec}(X, \tau_X^{\mathfrak{Z}})$ is left to the reader. ■

7.5. Locally Zariski quasi-pretopology. Let (X, τ_X) be a quasi-topological space. We call the quasi-pretopology τ_X *locally Zariski*, if there exists a conservative family of morphisms $\{U_i \xrightarrow{u_i} X \| \; i \in J\}$ such that all u_i belong to Σ^τ (cf. 1.5.1) and for every $i \in J$, the restriction τ_{U_i} of τ_X to U_i is the Zariski pretopology.

8. The center and the geometric center of a quasi-topological 'space'.

8.1. The center. For any 'space' X, we denote by $\mathcal{O}^{\mathfrak{c}}(X)$ the center of the category C_X, i.e. the (commutative) monoid of all endomorphisms of the identical functor Id_{C_X}. If C_X is an additive category, then $\mathcal{O}^{\mathfrak{c}}(X)$ is a commutative unital ring.

8.1.1. Example. If R is an associative unital ring and $X = \mathbf{Sp}(R)$, i.e. $C_X = R - mod$, then $\mathcal{O}^{\mathfrak{c}}(X)$ is naturally isomorphic to the center of the ring R.

8.1.2. Lemma. *The map $X \longmapsto \mathcal{O}^{\mathfrak{c}}(X)$ is functorial with respect to localizations.*

Proof. Let $C_Y \xrightarrow{u^*} C_X$ be a localization, and let $\xi \in \mathcal{O}^{\mathfrak{c}}(Y)$. Then $u^*(\xi) \in \mathcal{O}^{\mathfrak{c}}(X)$.

In fact, let Σ_u denote the family of all $s \in HomC_Y$ such that $u^*(s)$ is invertible. Any morphism $M_1 \xrightarrow{\varphi} M_n$ of the category C_X can be represented as the composition

$u^*(f_n)u^*(s_n)^{-1}\dots u^*(f_1)u^*(s_1)^{-1}$, where $M_i \xleftarrow{s_i} N_i \xrightarrow{f_i} M_{i+1}$ and $s_i \in \Sigma_u$ for all i. Since the diagram

$$\begin{array}{ccccc} M_i & \xleftarrow{s_i} & N_i & \xrightarrow{f_i} & M_{i+1} \\ {\scriptstyle \xi(M_i)}\downarrow & & {\scriptstyle \xi(N_i)}\downarrow & & \downarrow{\scriptstyle \xi(M_{i+1})} \\ M_i & \xleftarrow{s_i} & N_i & \xrightarrow{f_i} & M_{i+1} \end{array}$$

commutes, $u^*(\xi(M_{i+1}))\circ(u^*(f_i)u^*(s_i)^{-1}) = (u^*(f_i)u^*(s_i)^{-1})\circ u^*(\xi(M_i))$ for all i. Therefore,

$$u^*(\xi(M_n))\circ\varphi = u^*(\xi(M_n))\circ\big(u^*(f_n)u^*(s_n)^{-1}\dots\ u^*(f_1)u^*(s_1)^{-1}\big)$$

$$= \big(u^*(f_n)u^*(s_n)^{-1}\dots\ u^*(f_1)u^*(s_1)^{-1}\big)\circ u^*(\xi(M_1)) = \varphi\circ u^*(\xi(M_1)).$$

Hence the assertion. ■

8.1.3. The center of a quasi-topological 'space'. Let (X,τ) be a quasi-topological 'space' such that C_X is an additive category. By 8.1.2, the map $(U \xrightarrow{u} X) \longmapsto \mathcal{O}^{\mathfrak{c}}(U)$ defines a presheaf, $\mathcal{O}^{\mathfrak{c}}_X$, of commutative rings on (X,τ). We call this presheaf (or, ruther, the ringed 'space' $(X,\tau;\mathcal{O}^{\mathfrak{c}}_X)$) the *center* of (X,τ).

8.2. Spectra of a 'space' and the prime spectrum of its center.

8.2.1. Proposition. *Suppose C_X is an abelian category.*

(a) If X is local, then $\mathcal{O}^{\mathfrak{c}}(X)$ is a local ring.

(b) Let $\mathcal{P}$ be an element of $\mathbf{Spec}^1(X)$*, i.e. $\mathcal{P}$ is a thick subcategory of C_X such that $X/|\mathcal{P}|$ is local; and let $X/|\mathcal{P}| \xrightarrow{u_{\mathcal{P}}} X$ be the localization at $\mathcal{P}$. Suppose M is a quasi-final object of $C_X/\mathcal{P}$. Then $\mathfrak{p}_{\mathcal{P}} = \{\xi \in \mathcal{O}^{\mathfrak{c}}(X) \mid u^*_{\mathcal{P}}(\xi)(M) = 0\}$ is a prime ideal in the ring $\mathcal{O}^{\mathfrak{c}}(X)$. The ideal $\mathfrak{p}_{\mathcal{P}}$ does not depend on the choice of a quasi-final object M.*

(c) For every object M of Spec(X) and every $\xi \in \mathcal{O}^{\mathfrak{c}}(X)$, either $\xi(M) = 0$, or $\xi(M)$ is a monomorphism. In particular, the set $\{\xi \in \mathcal{O}^{\mathfrak{c}}(X) \mid \xi(M) = 0\}$ is a prime ideal in the ring $\mathcal{O}^{\mathfrak{c}}(X)$. This ideal depends only on the equivalence class of M.

Proof. (a) Let M be a quasi-final object of the category C_X. By the argument of [R, 3.2.1], the set $\mathfrak{m}_X = \{\xi \in \mathcal{O}^{\mathfrak{c}}(X) \mid \xi(M) = 0\}$ is the unique maximal ideal in the ring $\mathcal{O}^{\mathfrak{c}}(X)$. In other words, for any $\xi \in \mathcal{O}^{\mathfrak{c}}(X)$, either $\xi(M) = 0$, or ξ is an invertible element.

(b) By (the argument of) 8.1, the map $\xi \longmapsto u^*_{\mathcal{P}}(\xi)$ is a ring morphism $\mathcal{O}^{\mathfrak{c}}(X) \longrightarrow \mathcal{O}^{\mathfrak{c}}(X/|\mathcal{P}|)$. By (a), the set $\mathfrak{p}_{\mathcal{P}} = \{\xi \in \mathcal{O}^{\mathfrak{c}}(X) \mid u^*_{\mathcal{P}}(\xi)(M) = 0\}$ is the preimage of the unique maximal ideal of the local ring $\mathcal{O}^{\mathfrak{c}}(X/|\mathcal{P}|)$, hence it is a prime ideal.

(c) See [R, 7.1.2]. ■

8.2.2. Corollary. *Let C_X be an abelian category. There exist natural maps from the complete spectrum* $\mathbf{Spec}^1(X)$ *of the 'space' X to the prime spectrum, Spec$(\mathcal{O}^{\mathfrak{c}}_X)$, of the center of X and from the spectrum Spec(X) to Spec$(\mathcal{O}^{\mathfrak{c}}_X)$. These maps are related by the commutative diagram*

$$\begin{array}{ccc} \mathbf{Spec}(X) & \longrightarrow & \mathbf{Spec}^1(X) \\ \uparrow & & \downarrow \\ Spec(X) & \longrightarrow & Spec(\mathcal{O}^{\mathfrak{c}}_X) \end{array} \qquad (1)$$

The diagram (1) is functorial with respect to localizations.

Proof. The right vertical arrow and the lower horizontal arrow of (1) are provided by the assertions resp. 8.2.1(b) and 8.2.1(c). The left vertical arrow in (1) is the canonical map $M \longmapsto [M]$, and the upper horizontal arrow assigns to each element $[M]$ of $\mathbf{Spec}(X)$ the corresponding Serre subcategory $\langle M \rangle$. Verifying details is left to the reader. ■

8.3. The geometric centers of a quasi-topological 'space'. Let (X, τ_X) be a quasi-topological 'space'. Let $\mathbf{Spec}^\bullet(X, \tau_X)$ be any spectrum of this work realized as a subset of the complete spectrum $\mathbf{Spec}^1(X)$ and endowed with the topology induced by τ_X. With any subset V of $\mathbf{Spec}^1(X)$, we associate the quotient category $C_X/\mathbb{S}_V$, where $\mathbb{S}_V = \bigcap_{\mathcal{P} \in V} \mathcal{P}$. The map $\widetilde{\mathcal{O}}_X^{\mathfrak{c}}$ which assigns to every open set U of $\mathbf{Spec}^\bullet(X, \tau_X)$ the center of the category $C_X/\mathbb{S}_U$ is a presheaf of commutative rings on $\mathbf{Spec}^\bullet(X, \tau_X)$. We denote by $\mathcal{O}_X^{\mathfrak{c}}$ the associated sheaf and call the ringed topological space $(\mathbf{Spec}^\bullet(X, \tau_X), \mathcal{O}_X^{\mathfrak{c}})$ the *geometric center* of the 'space' (X, τ_X) associated with the spectrum $\mathbf{Spec}^\bullet(X, \tau_X)$.

In particular, we associate with the 'space' (X, τ_X) the ringed topological spaces $(\mathbf{Spec}_\wp(X, \tau_X), \mathcal{O}_X^{\mathfrak{c}})$ and $(\mathbf{Spec}(X, \tau_X), \mathcal{O}_X^{\mathfrak{c}})$ called respectively the *lower* (or *combinatorial*) and the *upper geometric center* of (X, τ_X).

8.4. Zariski geometric centers. The reconstruction of commutative schemes. Taking the Zariski pretopology, we assign to every $X \in Ob|\mathfrak{Ab}|^o$ the *lower* and the *upper Zariski geometric center* of X, resp. $(\mathbf{Spec}_\wp(X, \tau_X^{\mathfrak{Z}}), \mathcal{O}_X^{\mathfrak{c}})$ and $(\mathbf{Spec}(X, \tau_X^{\mathfrak{Z}}), \mathcal{O}_X^{\mathfrak{c}})$. If C_X is the category of quasi-coherent sheaves on a commutative scheme, then, by 7.4, these two spectra coincide.

8.4.1. Proposition. *Let C_X be the category of quasi-coherent sheaves on a scheme $\mathbf{X} = (\mathcal{X}, \mathcal{O})$. Then the Zariski geometric center of X is isomorphic to the scheme $\mathbf{X}$.*

Proof. (a) Suppose first that the space $\mathcal{X}$ is quasi-compact. By 7.4, the topological space $(\mathbf{Spec}(X, \tau_X^{\mathfrak{Z}}), \tau^{\mathfrak{Z}})$ is isomorphic to the underlying space $\mathcal{X}$ of the scheme $\mathbf{X}$. Let $\mathbf{U} = (\mathcal{U}, \mathcal{O}|_{\mathcal{U}})$ be an affine open subscheme of the scheme $\mathbf{X}$ corresponding to an element $U \longrightarrow X$ of an affine cover. Then the category $Qcoh_{\mathbf{U}}$ of quasi-coherent sheaves on $\mathbf{U}$ is equivalent to the category C_U. On the other hand, it is equivalent to the category of $\mathcal{O}(\mathcal{U})$-modules. The center of the category of $\mathcal{O}(\mathcal{U})$-modules is naturally isomorphic to the ring $\mathcal{O}(\mathcal{U})$. This shows that the presheaf which assigns to any affine open set $\mathcal{U}$ corresponding to an element $U \longrightarrow X$ of $\tau^{\mathfrak{Z}}$-cover of X the center of U (that is the center of the category C_U) is isomorphic to a presheaf which assigns to an affine open set $\mathcal{U}$ the ring $\mathcal{O}(\mathcal{U})$ of global sections of the structure sheaf $\mathcal{O}$ over $\mathcal{U}$. Therefore associated sheaves of these presheaves are isomorphic, hence the assertion.

(b) Consider now the general case. Let $\mathbf{X}$ be represented as a union (colimit) of a filtered system, $\{\mathbf{U}_\nu \mid \nu \in \mathfrak{J}\}$, of its quasi-compact open subschemes, $\mathbf{U}_\nu = (\mathcal{U}_\nu, \mathcal{O}|_{\mathcal{U}_\nu})$. Let C_{U_ν} be the category of quasi-coherent sheaves on the scheme $\mathbf{U}_\nu$. By 6.5.5.3,

$$\lim_{\nu \in \mathfrak{J}} \mathbf{Spec}_\star(U_\nu) \xrightarrow{\sim} \mathbf{Spec}_{\wp\star}(X_{\mathfrak{c}}, \tau_{X_{\mathfrak{c}}}^{\mathfrak{Z}}).$$

This isomorphism extends to an isomorphism of ringed spaces

$$\lim_{\nu \in \mathfrak{J}} (\mathbf{Spec}_\star(U_\nu), \mathcal{O}_{U_\nu}^{\mathfrak{c}}) \xrightarrow{\sim} (\mathbf{Spec}_{\wp\star}(X_{\mathfrak{c}}, \tau_{X_{\mathfrak{c}}}^{\mathfrak{Z}}), \mathcal{O}_X^{\mathfrak{c}}). \tag{1}$$

By (a), for any $\nu \in \mathfrak{J}$, there is a natural isomorphism

$$(\mathbf{Spec}(\mathcal{U}_\nu), \mathcal{O}^{\mathfrak{c}}_{U_\nu}) \xrightarrow{\sim} (\mathcal{U}_\nu, \mathcal{O}|_{\mathcal{U}_\nu}), \quad \text{and} \quad \lim_{\nu \in \mathfrak{J}} (\mathcal{U}_\nu, \mathcal{O}|_{\mathcal{U}_\nu})_\star \xrightarrow{\sim} (\mathcal{X}, \mathcal{O})_\star = \mathbf{X}_\star.$$

Here $(\mathcal{X}, \mathcal{O})_\star$ is a ringed space obtained by adjoining to $(\mathcal{X}, \mathcal{O})$ one point with zero ring. ■

9. Noncommutative schemes and their spectra.

9.1. Affine morphisms. A continuous morphism $X \xrightarrow{f} Y$ of 'spaces' is called *affine* if its direct image functor is conservative and has a right adjoint. It follows that the composition of affine morphisms is affine, and any isomorphism is affine.

9.1.1. Example: affine morphisms to $\mathbf{Sp}(R)$. For any associative ring R, the 'space' $\mathbf{Sp}(R)$ is defined by $C_{\mathbf{Sp}(R)} = R - mod$. The 'space' $\mathbf{Sp}(R)$ is called the *categoric spectrum* of the ring R. Let $R \xrightarrow{\varphi} A$ be an associative ring morphism. We denote by $\widetilde{\varphi}$ the morphism $\mathbf{Sp}(A) \longrightarrow \mathbf{Sp}(R)$ with the inverse image functor

$$R - mod \xrightarrow{\widetilde{\varphi}^*} A - mod, \quad M \longmapsto A \otimes_R M.$$

The morphism $\widetilde{\varphi}$ is affine. Its direct image functor, $\widetilde{\varphi}_*$, is the pull-back along φ; and the functor

$$R - mod \xrightarrow{\widetilde{\varphi}^!} A - mod, \quad M \longmapsto Hom_R(\widetilde{\varphi}_*(A), M),$$

is right adjoint to $\widetilde{\varphi}_*$. By [KR3, 6.6.1], any affine morphism $X \longrightarrow \mathbf{Sp}(R)$ is a composition of an isomorphism $X \xrightarrow{\sim} \mathbf{Sp}(A)$ for an associative ring A (hence the category C_X is equivalent to the category of left A-modules) and a morphism $\mathbf{Sp}(A) \xrightarrow{\widetilde{\varphi}} \mathbf{Sp}(R)$ corresponding to a unital ring morphism $R \xrightarrow{\varphi} A$.

The ring morphism $R \xrightarrow{\varphi} A$ can be chosen uniquely up to isomorphism. In fact, let $X \xrightarrow{f} \mathbf{Sp}(R)$ be an affine morphism with an inverse image functor f^*. Set $\mathcal{O} = f^*(R)$ and denote by φ the composition of the natural ring isomorphism $R^o \xrightarrow{\sim} Hom_R(R, R)$ and the morphism $Hom_R(R, R) \longrightarrow C_X(\mathcal{O}, \mathcal{O})$, $u \mapsto f^*(u)$. Here R^o is the ring opposite to R. Since an inverse image functor is determined uniquely up to isomorphism, the object $\mathcal{O}$ and the ring morphism $R \xrightarrow{\varphi^o} A = C_X(\mathcal{O}, \mathcal{O})^o$ are determined uniquely up to isomorphism.

9.2. Locally affine 'spaces' over a 'space'. Let (X, τ) be a quasi-topological 'space', and let $X \xrightarrow{f} Y$ be a morphism of 'spaces'. We call $(X, \tau; f)$ a *quasi-topological 'space' over Y.*

A morphism $U \xrightarrow{u} X$ is called *f-affine* if the composition $U \xrightarrow{fu} Y$ is affine. The morphism $U \xrightarrow{u} X$ is called an *affine open 'supspace'* of $(X, \tau; f)$ if fu is affine and u belongs to Σ^τ (i.e. u belongs to some τ-cover).

A quasi-topological 'space' $(X, \tau; f)$ over Y is said to be *locally affine* if the family of morphisms $\{U \xrightarrow{u} X \mid u \in \Sigma^\tau, \ fu \text{ is affine}\}$ is conservative.

Any conservative set, $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$, of affine open 'subspaces' is called an *affine cover* of the scheme $(X, \tau; f)$.

9.2.1. Locally affine 'spaces' over $\mathbf{Sp}(R)$. Let (X,τ) be a quasi-topological 'space' and R a unital associative ring. Let $X \xrightarrow{f} \mathbf{Sp}(R)$ be a locally affine morphism, and let $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ be an affine cover of (X,f). Set $\mathcal{O} = f^*(R)$ and $\mathcal{O}_{U_i} = u_i^*(\mathcal{O})$, $i \in J$. Since morphisms fu_i are affine and $\mathcal{O}_{U_i} = u_i^* f^*(R) \simeq (fu_i)^*(R)$, the object $\mathcal{O}_{U_i}$ is a projective cogenerator of finite type with a structure of a right R-module. The latter is a ring morphism $R^o \xrightarrow{\phi_i} C_{U_i}(\mathcal{O}_{U_i}, \mathcal{O}_{U_i})$.

9.3. Quasi-topological $\mathbb{Z}$-'spaces' and $\mathbb{Z}$-schemes. A $\mathbb{Z}$-*'space'* is a pair (X,f), where f is a continuous morphism $X \longrightarrow \mathbf{Sp}(\mathbb{Z})$. We denote by $\mathfrak{Esp}_{\mathbb{Z}}$ the full subcategory of $|\mathfrak{Ab}|^o/\mathbf{Sp}(\mathbb{Z})$ whose objects are $\mathbb{Z}$-'spaces'.

Let $\mathcal{E}sp_{\mathbb{Z}}$ denote the category whose objects are pairs $(X,\mathcal{O}_X)$, where X is a 'space' such that C_X is an abelian category and $\mathcal{O}_X$ is an object of C_X such that there exist coproducts of copies of $\mathcal{O}$. Morphisms $(X,\mathcal{O}_X) \longrightarrow (Y,\mathcal{O}_Y)$ are morphisms $X \xrightarrow{f} Y$ such that $f^*(\mathcal{O}_Y) \simeq \mathcal{O}_X$.

For every object $(X,\mathcal{O}_X)$ of the category $\mathcal{E}sp_{\mathbb{Z}}$, let $\Gamma\mathcal{O}_X$ denote the ring $C_X(\mathcal{O}_X,\mathcal{O}_X)^o$ opposite to the ring of endomorphisms of the object $\mathcal{O}_X$. By [BD, 6.6.23], the functor

$$C_X \xrightarrow{\Gamma_*} \Gamma\mathcal{O}_X - mod, \quad M \longmapsto C_X(\mathcal{O}_X, M),$$

has a left adjoint; i.e. it is a direct image functor of a continuous morphism.

9.3.1. Proposition. *The category $\mathfrak{Esp}_{\mathbb{Z}}$ of $\mathbb{Z}$-'spaces' is naturally equivalent to the category $\mathcal{E}sp_{\mathbb{Z}}$.*

Proof. Let Φ_* denote the map which assigns to each $\mathbb{Z}$-'space' (X,f) the pair $(X,\mathcal{O}_X)$, where $\mathcal{O}_X = f^*(\mathbb{Z})$ for some inverse image functor f^* of the morphism $X \xrightarrow{f} \mathbf{Sp}(\mathbb{Z})$, and to every morphism $(X,f) \xrightarrow{h} (Y,g)$ the morphism $(X,\mathcal{O}_X) \xrightarrow{h} (Y,\mathcal{O}_Y)$. Since the functor f^* has a right adjoint, it preserves 'small' coproducts; in particular it maps free $\mathbb{Z}$-modules to coproducts of copies of $\mathcal{O}_X = f^*(\mathbb{Z})$. Thus Φ_* is a functor from $\mathfrak{Esp}_{\mathbb{Z}}$ to $\mathcal{E}sp_{\mathbb{Z}}$. The quasi-inverse functor, Φ^*, assigns to each object $(X,\mathcal{O}_X)$ of the category $\mathcal{E}sp_{\mathbb{Z}}$ the pair (X,f), where f is the composition of the "global sections" morphism $X \longrightarrow \mathbf{Sp}(\Gamma\mathcal{O}_X)$ and the natural morphism $\mathbf{Sp}(\Gamma\mathcal{O}_X) \longrightarrow \mathbf{Sp}(\mathbb{Z})$. ■

9.3.2. Definitions. (a) A *quasi-topological $\mathbb{Z}$-'space'* is a triple $(X,\tau;f)$, where (X,τ) is a quasi-topological 'space' and (X,f) is a $\mathbb{Z}$-'space'. In other words, $(X,\tau;f)$ is a quasi-topological 'space' over $\mathbf{Sp}(\mathbb{Z})$ such that f is a continuouos morphism.

(b) A quasi-topological $\mathbb{Z}$-'space' $(X,\tau;f)$ is an *affine (Zariski) scheme* if the morphism f is affine and τ is the Zariski pretopology.

(c) A quasi-topological $\mathbb{Z}$-'space' $(X,\tau;f)$ is called a *(Zariski) $\mathbb{Z}$-scheme* if there exists a conservative cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ such that all u_i belong to Σ^τ, fu_i is an affine morphism, and the induced quasi-pretopology τ_{U_i} on U_i is the Zariski pretopology.

9.3.3. Proposition. *(a) For every quasi-topological $\mathbb{Z}$-'space' $(X,\tau;\mathcal{O}_X)$, the presheaf $\widetilde{\Gamma}\mathcal{O}_X$ defined by $\widetilde{\Gamma}\mathcal{O}_X(U \to X) = \Gamma\mathcal{O}_U = C_U(\mathcal{O}_U,\mathcal{O}_U)^o$ is a presheaf of algebras over the center of (X,τ).*

(b) A quasi-topological $\mathbb{Z}$-'space' $(X,\tau;f)$ is affine iff $\mathcal{O}_X = f^*(\mathbb{Z})$ is a projective generator of finite type and τ is the Zariski pretopology.

(c) A quasi-topological $\mathbb{Z}$-'space' $(X,\tau;f)$ is a $\mathbb{Z}$-scheme iff there exists a conservative family of morphisms $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ such that each u_i belongs to Σ^τ and $\mathcal{O}_{U_i} = u_i^*(\mathcal{O}_X)$ is a projective generator of finite type and the induced quasi-pretopology τ_{U_i} on U_i is the Zariski pretopology.

Proof. (a) In fact, for every arrow $U \xrightarrow{u} X$ of Σ^τ, we have a natural ring morphism $\mathcal{O}_U^{\mathfrak{c}} \longrightarrow \Gamma\mathcal{O}_U = C_U(\mathcal{O}_U,\mathcal{O}_U)^o$ taking values in the center of the ring $\Gamma\mathcal{O}_U$.

(b) A morphism $X \xrightarrow{f} \mathbf{Sp}(\mathbb{Z})$ is affine iff the associated with f canonical morphism $X \longrightarrow \mathbf{Sp}(\Gamma\mathcal{O}_X)$ is an isomorphism, or, equivalently, the functor

$$C_X \xrightarrow{\Gamma_*} \Gamma\mathcal{O}_X - mod, \quad M \longmapsto C_X(\mathcal{O}_X, M),$$

is an equivalence of categories. By a well-known result of Gabriel-Mitchell, this happens iff $\mathcal{O}_X$ is a projective generator of finite type.

(c) The assertion follows from (a) and the definition of a $\mathbb{Z}$-scheme. ■

9.4. Spectra of noncommutative schemes.

9.4.1. Spectra of affine schemes. Let (X,τ) be a quasi-topological 'space' such that $X = \mathbf{Sp}(R)$ for some associative unital ring R and τ is the Zariski topology. Then (X,τ) is a quasi-compact 'space' [R1]. Therefore, by 6.4.1(a), the combinatorial spectrum, $\mathbf{Spec}_\wp(X,\tau)$, of the 'space' (X,τ) coincides with $\mathbf{Spec}(X)$.

If the ring R is noncommutative, then the upper spectrum of (X,τ) might be larger than $\mathbf{Spec}(X)$.

9.4.2. The general case. Let $(X,\tau; X \xrightarrow{f} Y)$ be a scheme over Y with an affine cover $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$. Suppose that X has the property (sup), $\Sigma^\tau \subseteq \Sigma_\tau$ (cf. 1.5.1), and that every $U_i \to X,\ i \in J$, is a Serre localization. Then, by 6.4.1, the combinatorial spectrum of (X,τ) coincides with the spectrum $\mathbf{Spec}_\wp\mathfrak{U}$ of the cover $\mathfrak{U}$. If the cover $\mathfrak{U}$ is finite, then $\mathbf{Spec}_\wp(X,\tau) = \mathbf{Spec}(X)$.

By 5.2, the upper spectrum of (X,τ) equals to the union, $\bigcup_{i\in J}\mathbf{Spec}(U_i,\tau_{U_i})$, of the upper spectra of affine 'spaces' (U_i,τ_{U_i}). Thus, we have the following inclusions

$$\mathbf{Spec}(X) \subseteq \mathbf{Spec}_\wp(X,\tau) \subseteq \bigcup_{i\in J}\mathbf{Spec}(U_i) \subseteq \bigcup_{i\in J}\mathbf{Spec}(U_i,\tau_{U_i}) = \mathbf{Spec}(X,\tau) \subseteq \mathbf{Spec}^-(X).$$

9.4.3. Example: quasi-coherent sheaves of algebras, $\mathcal{D}$-Schemes. Let $\mathbf{X} = (\mathcal{X},\mathcal{O}_\mathcal{X})$ be a ringed topological space, and let $(\mathcal{A},\varphi)$ be a quasi-coherent sheaf of associative rings on $\mathbf{X}$; i.e. $\mathcal{A}$ is a sheaf of associative rings on the topological space $\mathcal{X}$ and φ is a morphism $\mathcal{O}_\mathcal{X} \longrightarrow \mathcal{A}$ of sheaves of rings which turns $\mathcal{A}$ into a sheaf of quasi-coherent left $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}$. Let $C_{\mathbf{Sp}(\mathcal{A})}$ denote the category of quasi-coherent $\mathcal{A}$-modules on $\mathbf{X}$. This defines the 'space' $\mathbf{Sp}(\mathcal{A})$. The morphism $\mathcal{O}_\mathcal{X} \xrightarrow{\varphi} \mathcal{A}$ induces two functors,

$$C_{\mathbf{Sp}(\mathcal{A})} \xrightarrow{\widetilde{\varphi}_*} C_{\mathbf{Sp}(\mathcal{O}_\mathcal{X})} = Qcoh(\mathbf{X}) \xrightarrow{\widetilde{\varphi}^*} C_{\mathbf{Sp}(\mathcal{A})}$$

which are interpreted as resp. direct and inverse image functors of a morphism of 'spaces' $\mathbf{Sp}(\mathcal{A}) \xrightarrow{\widetilde{\varphi}} \mathbf{Sp}(\mathcal{O}_{\mathcal{X}})$. If $\mathbf{X}$ is a scheme, then every affine cover of $\mathbf{X}$ induces an affine cover of $\mathbf{Sp}(\mathcal{A})$. Therefore, $\mathbf{Sp}(\mathcal{A})$ is a scheme.

Let τ be the Zariski pretopology on $\mathbf{Sp}(\mathcal{A})$. Let $\mathfrak{U} = \{\mathcal{U}_i \to \mathcal{X} \mid i \in J\}$ be an affine cover of the scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Then the upper spectrum of $(\mathbf{Sp}(\mathcal{A}), \tau)$ is the union $\bigcup_{i \in J} \mathbf{Spec}(\mathbf{Sp}(\mathcal{A}(\mathcal{U}_i)), \tau_{\mathcal{U}_i})$ of the upper spectra of the schemes $(\mathbf{Sp}(\mathcal{A}(\mathcal{U}_i)), \tau_{\mathcal{U}_i})$.

Suppose, for example, that $\mathbf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a smooth scheme over $Spec(k)$, where k is a field of zero characteristic; and let $\mathcal{A} = \mathcal{D}_{\mathbf{X}}$ be the sheaf of differential operators on $\mathbf{X}$. Since the scheme $\mathbf{X}$ is smooth, one can choose an affine open cover formed by affine spaces. Let $\mathcal{U} = \{\mathcal{U}_i \to \mathcal{X} \mid i \in J\}$ be such a cover. Then all algebras $\mathcal{A}(\mathcal{U}_i)$ are isomorphic to the Weyl algebra of the rank equal to the dimension of $\mathcal{U}_i$. In particular, they are simple, hence the Zariski pretopology on $\mathbf{Sp}(\mathcal{A}(\mathcal{U}_i)$ is trivial. Therefore, the upper spectrum of the D-scheme $(\mathbf{X}, \mathcal{D}_{\mathbf{X}})$ is the union of the spectra of Weyl algebras.

Other examples of this work are noncommutative versions of quasi-affine and projective schemes.

9.5. The cone of a non-unital ring. Let R_0 be a unital associative ring, and let R_+ be an associative ring, non-unital in general, in the category of R_0-bimodules; i.e. R_+ is endowed with an R_0-bimodule morphism $R_+ \otimes_{R_0} R_+ \xrightarrow{m} R_+$ satisfying the associativity condition. Let $R = R_0 \oplus R_+$ denote the augmented ring described by this data. Let $\mathcal{T}_{R_+}$ denote the full subcategory of the category $R - mod$ whose objects are all R-modules annihilated by R_+. Let $\mathcal{T}_{R_+}^-$ be the Serre subcategory (that is a full subcategory closed by taking subquotients, extensions, and arbitrary direct sums) of the category $R - mod$ spanned by $\mathcal{T}_{R_+}$.

We define the *cone of* R_+ by taking as $C_{\mathbf{Cone}(R_+)}$ the quotient category $R - mod/\mathcal{T}_{R_+}^-$. The localization functor $R - mod \xrightarrow{u^*} R - mod/\mathcal{T}_{R_+}^-$ is an inverse image functor of a morphism of 'spaces' $\mathbf{Cone}(R_+) \xrightarrow{u} \mathbf{Sp}(R)$. The functor u^* has a (necessarily fully faithful) right adjoint, i.e. the morphism u is continuous. If R_+ is a unital ring, then u is an isomorphism (see [KR3, C3.2.1]). The composition of the morphism u with the canonical affine morphism $\mathbf{Sp}(R) \longrightarrow \mathbf{Sp}(R_0)$ is a continuous morphism $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Sp}(R_0)$. Its direct image functor is (regarded as) the *global sections functor*.

It follows from [R, 3.6.5.2] that the Zariski pretopology on $\mathbf{Cone}(R_+)$ is quasi-compact if R_+ is an R_0-module of finite type. Therefore, by 6.4.1(a),

$$\mathbf{Spec}_{\wp}^-(\mathbf{Cone}(R_+), \tau^3) = \mathbf{Spec}(\mathbf{Cone}(R_+)).$$

In the general case, R_+ is the union of a filtered family, $\{\alpha_i \mid i \in J\}$, of finitely generated two-sided ideals in R. Then $\mathbf{Cone}(R_+) = sup_{i \in J}\mathbf{Cone}(\alpha_i)$, hence, by 6.5.5.3 and 6.4.1,

$$\mathbf{Spec}_{\wp\star}^-(\mathbf{Cone}(R_+), \tau^3) \simeq \lim_{i \in J} \mathbf{Spec}_\star(\mathbf{Cone}(\alpha_i)).$$

Note that, in general, $\mathbf{Cone}(R_+)$ is not a scheme.

9.6. $\mathbf{Proj}_{\mathcal{G}}$. Let $\mathcal{G}$ be a monoid and $R = R_0 \oplus R_+$ a $\mathcal{G}$-graded ring with zero component R_0. Then we have the category $gr_{\mathcal{G}}R - mod$ of $\mathcal{G}$-graded R-modules and its full subcategory $gr_{\mathcal{G}}\mathcal{T}_{R_+} = \mathcal{T}_{R_+} \cap gr_{\mathcal{G}}R - mod$ whose objects are graded modules annihilated by the ideal R_+. We define the 'space' $\mathbf{Proj}_{\mathcal{G}}(R)$ by setting

$$C_{\mathbf{Proj}_{\mathcal{G}}(R)} = gr_{\mathcal{G}}R - mod/gr_{\mathcal{G}}\mathcal{T}^-_{R_+}.$$

Here $gr_{\mathcal{G}}\mathcal{T}^-_{R_+}$ is the Serre subcategory of the category $gr_{\mathcal{G}}R - mod$ spanned by $gr_{\mathcal{G}}\mathcal{T}_{R_+}$.

We show below that $gr_{\mathcal{G}}\mathcal{T}^-_{R_+} = gr_{\mathcal{G}}R - mod \cap \mathcal{T}^-_{R_+}$ (see 9.8, 9.8.1). Therefore, there is a canonical projection

$$\mathbf{Cone}(R_+) \xrightarrow{\mathfrak{p}} \mathbf{Proj}_{\mathcal{G}}(R).$$

The localization functor $gr_{\mathcal{G}}R - mod \longrightarrow C_{\mathbf{Proj}_{\mathcal{G}}(R_+)}$ is an inverse image functor of a continuous morphism $\mathbf{Proj}_{\mathcal{G}}(R) \xrightarrow{\mathfrak{v}} \mathbf{Sp}_{\mathcal{G}}(R)$, where $C_{\mathbf{Sp}_{\mathcal{G}}(R)} = gr_{\mathcal{G}}R - mod$. The composition $\mathbf{Proj}_{\mathcal{G}}(R) \xrightarrow{\mathfrak{v}} \mathbf{Sp}(R_0)$ of the morphism $\mathfrak{v}$ with the canonical morphism $\mathbf{Sp}_{\mathcal{G}}(R) \xrightarrow{\phi} \mathbf{Sp}(R_0)$ defines $\mathbf{Proj}_{\mathcal{G}}(R)$ as a 'space' over $\mathbf{Sp}(R_0)$. Its direct image functor is called the *global sections functor*.

9.6.1. Cone and Proj of a $\mathbb{Z}_+$-graded ring. Let $R = \oplus_{n\geq 0}R_n$ be a $\mathbb{Z}_+$-graded ring, $R_+ = \oplus_{n\geq 1}R_n$ its 'irrelevant' ideal. Thus, we have the *cone*, $\mathbf{Cone}(R_+)$, of the ring R_+, and $\mathbf{Proj}(R) = \mathbf{Proj}_{\mathbb{Z}}(R)$, and a canonical morphism $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Proj}(R)$.

9.6.2. Example: the base affine 'space' and the flag variety of a reductive Lie algebra. Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ and $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. Let $\mathcal{G}$ be the group of integral weights of $\mathfrak{g}$ and $\mathcal{G}_+$ the semigroup of nonnegative integral weights. Let $R = \oplus_{\lambda\in\mathcal{G}_+}R_\lambda$, where R_λ is the vector space of the (canonical) irreducible finite dimensional representation with the highest weight λ. The module R is a $\mathcal{G}$-graded algebra with the multiplication determined by the projections $R_\lambda \otimes R_\nu \longrightarrow R_{\lambda+\nu}$, for all $\lambda, \nu \in \mathcal{G}_+$. It is well known that the algebra R is isomorphic to the algebra of regular functions on the *base affine space* of $\mathfrak{g}$. Recall that G/U, where G is a connected simply connected algebraic group with the Lie algebra $\mathfrak{g}$, and U is its maximal unipotent subgroup.

The category $C_{\mathbf{Cone}(R)}$ is equivalent to the category of quasi-coherent sheaves on the base affine space Y of the Lie algebra $\mathfrak{g}$. The category $Proj_{\mathcal{G}}(R)$ is equivalent to the category of quasi-coherent sheaves on the flag variety of $\mathfrak{g}$.

9.6.3. The quantized base affine 'space' and quantized flag variety of a semisimple Lie algebra. Let now $\mathfrak{g}$ be a semisimple Lie algebra over a field k of zero characteristic, and let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$. Define the $\mathcal{G}$-graded algebra $R = \oplus_{\lambda\in\mathcal{G}_+}R_\lambda$ the same way as above. This time, however, the algebra R is not commutative. Following the classical example (and identifying spaces with categories of quasi-coherent sheaves on them), we call $\mathbf{Cone}(R)$ *the quantum base affine 'space'* and $\mathbf{Proj}_{\mathcal{G}}(R)$ the *quantum flag variety* of $\mathfrak{g}$.

9.6.4. Canonical affine covers of the base affine 'space' and the flag variety. Let W be the Weyl group of the Lie algebra $\mathfrak{g}$. Fix a $w \in W$. For any $\lambda \in \mathcal{G}_+$, choose a nonzero w-extremal vector $e^\lambda_{w\lambda}$ generating the one dimensional vector subspace of R_λ

formed by the vectors of the weight $w\lambda$. Set $S_w = \{k^* e^\lambda_{w\lambda} | \lambda \in \mathcal{G}_+\}$. It follows from the Weyl character formula that $e^\lambda_{w\lambda} e^\mu_{w\mu} \in k^* e^{\lambda+\mu}_{w(\lambda+\mu)}$. Hence S_w is a multiplicative set. It was proved by Joseph [Jo] that S_w is a left and right Ore subset in R. The Ore sets $\{S_w | w \in W\}$ determine a conservative family of affine localizations

$$\mathbf{Sp}(S_w^{-1}R) \longrightarrow \mathbf{Cone}(R), \quad w \in W, \tag{4}$$

of the quantum base affine 'space' and a conservative family of affine localizations

$$\mathbf{Sp}_{\mathcal{G}}(S_w^{-1}R) \longrightarrow \mathbf{Proj}_{\mathcal{G}}(R), \quad w \in W,$$

of the quantum flag variety. We claim that the category $gr_{\mathcal{G}} S_w^{-1}R - mod$ is naturally equivalent to $(S_w^{-1}R)_0 - mod$. By 1.5, it suffices to verify that the canonical functor $gr_{\mathcal{G}} S_w^{-1}R - mod \longrightarrow (S_w^{-1}R)_0 - mod$ which assigns to every graded $S_w^{-1}R$-module its zero component is faithful; i.e. the zero component of every nonzero $\mathcal{G}$-graded $S_w^{-1}R$-module is nonzero. This is, really, the case, because if z is a nonzero element of λ-component of a $\mathcal{G}$-graded $S_w^{-1}R$-module, then $(e^\lambda_{w\lambda})^{-1}z$ is a nonzero element of the zero component of this module.

9.7. Actions. Let $\mathcal{G}$ be a monoid. An action of $\mathcal{G}$ on a 'space' X is a monoidal functor $\mathcal{G} \xrightarrow{\mathfrak{L}} \widetilde{End}(C_X)$. Here $\mathcal{G}$ is viewed as a discrete monoidal category and $\widetilde{End}(C_X)$ denote the (strict) monoidal category of endofunctors $C_X \longrightarrow C_X$; i.e. $\widetilde{End}(C_X) = (End(C_X), \circ)$.

9.7.1. Examples. (a) Let R_+ be a (non-unital in general) $\mathcal{G}$-graded R_0-ring (cf. 9.5); and let $C_Y = gr_{\mathcal{G}} R_+ - mod_1$. For any $\mathcal{G}$-graded R_+-module $N = \oplus_{\nu \in \mathcal{G}} N_\nu$ and any $\gamma \in \mathcal{G}$, we denote by $N[\gamma]$ the $\mathcal{G}$-graded R_+-module defined by $N[\gamma]_\sigma = N_{\sigma\gamma}$. This defines a *strict* action of $\mathcal{G}$ on the 'space' Y. Here *strict* means that the monoidal functor $\mathcal{G} \xrightarrow{\mathfrak{L}} \widetilde{End}(C_X)$ is strict, that is $N[\gamma_1\gamma_2] = (N[\gamma_2])[\gamma_1]$ for all N.

(b) The action of $\mathcal{G}$ on the 'space' Y in (a) (i.e. on the category $gr_{\mathcal{G}} R_+ - mod_1$), induces an action of $\mathcal{G}$ on $\mathbf{Proj}_{\mathcal{G}}(R_+)$.

9.8. Proposition. *Let $\mathbb{T}$ be a $\mathcal{G}$-stable, topologizing subcategory of $gr_{\mathcal{G}} R_+ - mod_1$, and let $\widetilde{\mathbb{T}}$ denote the image of $\mathbb{T}$ in $R_+ - mod_1$. Then ${\pi^*}^{-1}(\widetilde{\mathbb{T}}^-) = \mathbb{T}^-$.*

Proof. (a) Since the functor (1) is exact, the preimage, ${\pi^*}^{-1}(\widetilde{\mathbb{T}}^-)$, of the Serre subcategory $\widetilde{\mathbb{T}}^-$ is a Serre subcategory of the category $gr_{\mathcal{G}} R_+ - mod_1$. The inclusion ${\pi^*}^{-1}(\widetilde{\mathbb{T}}^-) \subseteq \mathbb{T}^-$ is equivalent to that every nonzero object of ${\pi^*}^{-1}(\widetilde{\mathbb{T}}^-)$ has a nonzero subobject which belongs to $\mathbb{T}$; or, what is the same, for any nonzero object, N, of ${\pi^*}^{-1}(\widetilde{\mathbb{T}}^-)$, there exists a nonzero morphism $L \xrightarrow{g} N$, with $L \in Ob\mathbb{T}$. We can and will assume that L is generated by one of its homogeneous components. Then $R_+ - mod_1(L, N)$ is a $\mathcal{G}$-graded $\mathbb{Z}$-module, and some of homogeneous components of the morphism g are nonzero. Replacing the module L by the module $L[\gamma]$ for an appropriate $\gamma \in \mathcal{G}$, we can assume that the homogeneous component of g of zero degree is nonzero. Thus, there exists a nonzero morphism $L[\gamma] \longrightarrow N$ of graded R_+-modules. Since the subcategory $\mathbb{T}$ is stable under the action of $\mathcal{G}$, the object $L[\gamma]$ belongs to $\mathbb{T}$.

(b) Every object, M, of $\mathbb{T}^-$ has a filtration, $\{M_i \mid i \geq 0\}$ such that $M_i = sup(M_j|j < i)$, if i is a limit ordinal, and M_{i+1}/M_i belongs to $\mathbb{T}$. But, this implies that M is an object of $\widetilde{\mathbb{T}}^-$; i.e. we have the inverse inclusion, $\mathbb{T}^- \subseteq {\pi^*}^{-1}(\widetilde{\mathbb{T}}^-)$. ■

9.8.1. Corollary. $gr_{\mathcal{G}}\mathcal{T}^-_{R_+} = {\pi^*}^{-1}(\mathcal{T}^-_{R_+})$.

Proof. Set $\mathbb{T} = gr_{\mathcal{G}}\mathcal{T}_{R_+}$. Then $\widetilde{\mathbb{T}}^-$ coincides with $\mathcal{T}^-_{R_+}$, hence the assertion. ■

9.8.2. Corollary. *The forgetful functor (1) induces a faithful exact functor*

$$C_{\mathbf{Proj}_{\mathcal{G}}(\mathbf{R}_+)} \xrightarrow{\mathfrak{p}^*} C_{\mathbf{Cone}(\mathbf{R}_+)} \tag{2}$$

Proof. By 9.8.1(b), $gr_{\mathcal{G}}\mathcal{T}^-_{R_+} = {\pi^*}^{-1}(\mathcal{T}^-_{R_+})$, where $gr_{\mathcal{G}}R_+ - mod_1 \xrightarrow{\pi^*} R_+ - mod_1$ is the forgetful functor. The functor ${\pi^*}^{-1}$ induces a faithful functor between quotient categories

$$gr_{\mathcal{G}}R_+ - mod_1/gr_{\mathcal{G}}\mathcal{T}^-_{R_+} \longrightarrow R_+ - mod_1/\mathcal{T}^-_{R_+}.$$

This functor is exact because the inclusion functor $gr_{\mathcal{G}}R_+ - mod_1 \longrightarrow R_+ - mod_1$ is exact. Hence the assertion. ■

The functor (2) is regarded as an inverse image functor of a morphism ('projection') $\mathbf{Cone}(R_+) \xrightarrow{\mathfrak{p}} \mathbf{Proj}_{\mathcal{G}}(R_+)$.

9.9. $\mathcal{G}$-Spectrum. Fix an action of a monoid $\mathcal{G}$ on a 'space' X. For every $M \in ObC_X$, we denote by $[M]_{\mathcal{G}}$ the smallest $\mathcal{G}$-stable topologizing subcategory of C_X containing M. Let $\langle M\rangle_{\mathcal{G}}$ be the full subcategory of C_X whose objects are all $N \in ObC_X$ such that M does not belong to $[N]_{\mathcal{G}}$. It follows from this definition that the subcategory $\langle M\rangle_{\mathcal{G}}$ is $\mathcal{G}$-stable.

We write $N \overset{\mathcal{G}}{\succ} M$ if M is contained in $[N]_{\mathcal{G}}$. It is easy to see that $N \overset{\mathcal{G}}{\succ} M \Leftrightarrow [M]_{\mathcal{G}} \subseteq [N]_{\mathcal{G}} \Leftrightarrow \langle M\rangle_{\mathcal{G}} \subseteq \langle N\rangle_{\mathcal{G}}$.

We denote by $Spec_{\mathcal{G}}(X)$ the preorder defined as follows. Objects of $Spec_{\mathcal{G}}(X)$ are nonzero objects P of C_X such that if there exists a non-trivial morphism $N \longrightarrow M$, then $N \overset{\mathcal{G}}{\succ} M$. The preorder is given by $\overset{\mathcal{G}}{\succ}$.

We denote by $\mathbf{Spec}_{\mathcal{G}}(X)$ the preorder whose objects are $\mathcal{G}$-stable topologizing subcategories $[M]_{\mathcal{G}}$, $M \in Spec_{\mathcal{G}}(X)$, and the preorder is the inverse inclusion, $\supseteq$. The map $M \longmapsto [M]_{\mathcal{G}}$ induces an epimorphism of preorders $Spec_{\mathcal{G}}(X) \longrightarrow \mathbf{Spec}_{\mathcal{G}}(X)$.

It follows that the relation $N \succ M$ implies that $N \overset{\mathcal{G}}{\succ} M$. In particular, there is a natural embedding $Spec(X) \longrightarrow Spec_{\mathcal{G}}(X)$. This embedding induces a morphism $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}_{\mathcal{G}}(X)$ such that the diagram

$$\begin{array}{ccc} Spec(X) & \longrightarrow & Spec_{\mathcal{G}}(X) \\ \downarrow & & \downarrow \\ \mathbf{Spec}(X) & \longrightarrow & \mathbf{Spec}_{\mathcal{G}}(X) \end{array}$$

commutes.

9.9.1. Proposition. *Let a monoid $\mathcal{G}$ act on a 'space' X. Suppose that there exists a family $\{\mathbb{T}_i \mid i \in J\}$ of $\mathcal{G}$-stable Serre subcategories of the category C_X such that*

(a) $\bigcap_{i \in J} \mathbb{T}_i = 0$;

(b) every $\gamma \in \mathcal{G}$ induces a trivial (that is isomorphic to the identical functor) autoequivalence of $C_X/\mathbb{T}_i$ for all $i \in J$.

Then the canonical map $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}_{\mathcal{G}}(X)$ *is an isomorphism.*

Proof. Let U_i denote the 'space' defined by $C_{U_i} = C_X/\mathbb{T}_i$. The condition (a) means that every localization functor $C_X \longrightarrow C_{U_i}$ factors through a localization functor $C_X \xrightarrow{q_i^*} C_{U_i}$, and the family of localization functors $\{q_i^* \mid i \in J\}$ is conservative. The condition (b) implies that $\mathcal{G}$ acts trivially on each category C_{U_i}, hence it acts trivially on $\mathbf{Spec}(U_i)$ for every $i \in J$. This means that $\mathbf{Spec}(U_i) = \mathbf{Spec}_{\mathcal{G}}(U_i)$. By 2.4.2, there is a natural embedding $\mathbf{Spec}(X) \hookrightarrow \bigcup_{i \in J} \mathbf{Spec}(U_i)$, where each $\mathbf{Spec}(U_i)$ is identified with the corresponding subset of the complete spectrum $\mathbf{Spec}^1(X)$. Therefore, $\mathcal{G}$ acts trivially on $\mathbf{Spec}(X)$, which implies that $\mathbf{Spec}(X) = \mathbf{Spec}_{\mathcal{G}}(X)$. ■

9.9.1.2. Note. It is useful to see what is going on with representatives of the points of the spectrum, i.e. objects of $Spec(X)$. Let M be an object of $Spec(X)$ and let γ be an arbitrary element of $\mathcal{G}$. Since the family of localizations is conservative, there exist $i \in J$ such that $q_i^*(M) \neq 0$. Since M belongs to $Spec(X)$ this means that M is $\mathbb{T}_i$-torsion free and $q_i^*(M)$ is an object of $Spec(U_i)$ (see the argument of 2.4.2). Since $q_i^*(M[\gamma]) = q_i^*(M)[\gamma]$, the condition (b) implies that $q_i^*(M) \simeq q_i^*(M[\gamma])$. In particular, $[q_i^*(M)] = [q_i^*(M[\gamma])]$.

9.10. Proposition. *An object M of $C_{\mathbf{Proj}_{\mathcal{G}}(\mathbf{R}_+)}$ belongs to $Spec_{\mathcal{G}}(\mathbf{Proj}_{\mathcal{G}}(R_+))$ iff its image in $C_{\mathbf{Cone}(\mathbf{R}_+)}$ belongs to $Spec(\mathbf{Cone}(\mathbf{R}_+))$.*

Proof. (a) Let M be a graded $gr\mathcal{T}_{R_+}$-torsion free R_+-module such that its image in $C_{\mathbf{Cone}(\mathbf{R}_+)}$ belongs to $Spec(\mathbf{Cone}(\mathbf{R}_+))$. Let $\mathbb{T}$ be any $\mathcal{G}$-stable Serre subcategory of $grR_+ - mod_1$ (i.e. $\mathbb{T}^- = \mathbb{T}$) which contains the subcategory $gr\mathcal{T}_{R_+}$ and does not contain the object M. Let $\widetilde{\mathbb{T}}$ denote the image of $\mathbb{T}$ in $R_+ - mod_1$. It follows that $\widetilde{\mathbb{T}}^-$ contains the Serre subcategory $\mathcal{T}_{R_+}^-$. By 9.8(b), $\mathbb{T}$ coincides with the preimage of $\widetilde{\mathbb{T}}^-$ in $grR_+ - mod_1$. Therefore, the Serre subcategory $\widetilde{\mathbb{T}}^-$ does not contain the image $\widetilde{M}$ of M in $R_+ - mod_1$ ($\widetilde{M}$ coincides with the R_+-module M with forgotten grading). This means that $\widetilde{\mathbb{T}} \subseteq \langle \widetilde{M} \rangle_{\mathcal{T}_{R_+}}$, hence $\mathbb{T} \subseteq \pi^{*^{-1}}(\langle \widetilde{M} \rangle_{\mathcal{T}_{R_+}})$. Here $\langle \widetilde{M} \rangle_{\mathcal{T}_{R_+}}$ denotes the union of all Serre subcategories of $R_+ - mod_1$ which contain the subcategory $\mathcal{T}_{R_+}$, but do not contain the object $\widetilde{M}$. Notice that the subcategory $\langle \widetilde{M} \rangle_{\mathcal{T}_{R_+}}$ is the preimage in $R_+ - mod_1$ of the subcategory $\langle \widetilde{M}' \rangle$, where $\widetilde{M}'$ denote the image in $C_{\mathbf{Cone}(\mathbf{R}_+)}$ of the object $\widetilde{M}$. By hypothesis, the object $\widetilde{M}'$ belongs to $Spec(\mathbf{Cone}(\mathbf{R}_+))$ which means precisely that $\langle \widetilde{M}' \rangle$ is a Serre subcategory of $C_{\mathbf{Cone}(\mathbf{R}_+)}$. Therefore $\langle \widetilde{M} \rangle_{\mathcal{T}_{R_+}}$ is a Serre subcategory of $R_+ - mod_1$. The latter implies that the preimage, $\pi^{*^{-1}}(\widetilde{M} \rangle_{\mathcal{T}_{R_+}})$, of $\langle \widetilde{M} \rangle_{\mathcal{T}_{R_+}}$ is the largest $\mathcal{G}$-stable Serre subcategory of $grR_+ - mod_1$ containing $gr\mathcal{T}_{R_+}$ which does not contain the object M. Therefore, the image of M in $C_{\mathbf{Proj}_{\mathcal{G}}(\mathbf{R}_+)}$ belongs to $Spec_{\mathcal{G}}(\mathbf{Proj}_{\mathcal{G}}(\mathbf{R}_+))$.

(b) Let M be a graded $gr\mathcal{T}_{R_+}$-torsion free R_+-module such that its image in $C_{\mathbf{Proj}_\mathcal{G}(\mathbf{R}_+)}$ belongs to $Spec_\mathcal{G}(\mathbf{Proj}_\mathcal{G}(\mathbf{R}_+))$. Since by 9.8.2, the functor $C_{\mathbf{Proj}_\mathcal{G}(\mathbf{R}_+)} \xrightarrow{\mathfrak{p}^*} C_{\mathbf{Cone}(\mathbf{R}_+)}$ is faithful, the image $\widetilde{M}$ of M in $R_+ - mod_1$ is $\mathcal{T}_{R_+}$-torsion free. Let $N \longrightarrow \widetilde{M}$ be a nonzero monomorphism. Because N is a nonzero R_+-module, there exists a nonzero monomorphism $\widetilde{L} \longrightarrow N$ for some graded R_+-module L generated by one of its homogeneous components. Since the composition $\widetilde{L} \longrightarrow M$ is a nonzero (mono)morphism, one of its graded components is nonzero; i.e. there exists a nonzero morphism $L[\gamma] \xrightarrow{g} M$ of graded R_+-modules. Since M belongs to $Spec_\mathcal{G}(\mathbf{Proj}_\mathcal{G}(\mathbf{R}_+))$, there exists a graded submodule M' of M such that $L[\gamma] \succ M'$ (i.e. $M' \in Ob[L[\gamma]]$) and M/M' belongs to $gr\mathcal{T}^-_{R_+}$. Since the forgetful functor $gr_\mathcal{G}R_+ - mod_1 \longrightarrow R_+ - mod_1$ is exact, $\widetilde{L[\gamma]} \succ \widetilde{M'}$, and the quotient $\widetilde{M}/\widetilde{M'} = \widetilde{M/M'}$ is an object of $\mathcal{T}^-_{R_+}$. Let q^* denote the localization functor $R_+ - mod_1 \longrightarrow C_{\mathbf{Cone}(\mathbf{R}_+)}$. Since the functor q^* is exact, $q^*(L) \longrightarrow N$ is a monomorphism; in particular, $q^*(N) \succ q^*(\widetilde{L})$. Notice that $\widetilde{L} \simeq \widetilde{L}[\gamma]$ and $q^*(\widetilde{M'}) \simeq q^*(\widetilde{M})$. Thus, $q^*(N) \succ q^*(\widetilde{L[\gamma]}) \succ q^*(\widetilde{M'}) \simeq q^*(\widetilde{M})$. This proves that the image, $q^*(\widetilde{M})$, of M in $C_{\mathbf{Cone}(\mathbf{R}_+)}$ belongs to $Spec(\mathbf{Cone}(\mathbf{R}_+))$. ■

9.10.1. Corollary. *The canonical morphism* $\mathbf{Cone}(R_+) \xrightarrow{\mathfrak{p}} \mathbf{Proj}_\mathcal{G}(R_+)$ *induces a map*

$$\mathbf{Spec}_\mathcal{G}(\mathbf{Proj}_\mathcal{G}(\mathbf{R}_+)) \longrightarrow \mathbf{Spec}(\mathbf{Cone}(\mathbf{R}_+)). \qquad (1)$$

Proof. By 9.10, the inverse image functor of the morphism $\mathfrak{p}$ induces a functor

$$Spec_\mathcal{G}(\mathbf{Proj}_\mathcal{G}(\mathbf{R}_+)) \longrightarrow Spec(\mathbf{Cone}(\mathbf{R}_+))$$

which gives a rise to the map (1). ■

9.10.2. Proposition. *Suppose that there exists a family* $\{\mathbb{T}_i \mid i \in J\}$ *of* $\mathcal{G}$*-stable Serre subcategories of the category* $gr_\mathcal{G}R_+ - mod_1$ *such that*

(a) $\bigcap_{i\in J} \mathbb{T}_i = gr_\mathcal{G}\mathcal{T}^-_{R_+}$*;*

(b) every $\gamma \in \mathcal{G}$ *induces a trivial (that is isomorphic to the identical functor) autoequivalence of* $gr_\mathcal{G}R_+ - mod_1/\mathbb{T}_i$ *for all* $i \in J$*. Then* $\mathbf{Spec}_\mathcal{G}(\mathbf{Proj}_\mathcal{G}(R_+) = \mathbf{Spec}(\mathbf{Proj}_\mathcal{G}(R_+)$*, and the canonical map (1) is injective and identifies* $\mathbf{Spec}(\mathbf{Proj}_\mathcal{G}(\mathbf{R}_+))$ *with the subset of all* $[M]$ *such that* M *is a graded* R_+*-module which belongs to* $Spec(\mathbf{Cone}(\mathbf{R}_+))$*.*

Proof. The assertion is a corollary of 9.9.1. ■

9.10.3. Proposition. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra over a field of zero characteristic, and let* $R = \oplus_{\lambda\in\mathcal{G}_+} R_\lambda$ *is the algebra of functions on the base affine 'space' of* $\mathfrak{g}$ *(cf. 9.6.3). Then*

$$\mathbf{Spec}(\mathbf{Proj}_\mathcal{G}(R)) \simeq \mathbf{Spec}_\mathcal{G}(\mathbf{Proj}_\mathcal{G}(R)) \simeq \{[M] \mid M \in Spec(\mathbf{Cone}(R)) \bigcap gr_\mathcal{G}R - mod\}.$$

Proof. The canonical cover $\{\mathbf{Sp}_\mathcal{G}(S_w^{-1}R) \longrightarrow \mathbf{Proj}_\mathcal{G}(R) \mid w \in W\}$ (cf. 9.6.4) satisfies the conditions of 9.9.1, hence the assertion. ■

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