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RUSSELL AND FREGE ON THE LOGIC OF FUNCTIONS

ABSTRACT: I compare Russell's theory of mathematical functions, the "descriptive functions" from *Principia Mathematica* ∗30, with Frege's well known account of functions as "unsaturated" entities. Russell analyses functional terms with propositional functions and the theory of definite descriptions. This is the primary technical role of the theory of descriptions in *PM*. In *Principles of Mathematics* and some unpublished writings from before 1905, Russell offered explicit criticisms of Frege's account of functions. Consequenly, the theory of descriptions in "On Denoting" can be seen as a crucial part of Russell's larger logicist reduction of mathematics, as well as an excursion into the theory of reference.

Russell's theory of definite descriptions, with its accompanying notions of scope and contextual definition, is justifiably still a leading theory in the philosophy of language, over one hundred years since it was first published in "On Denoting" in 1905. This theory was certainly an early paradigm of analytic philosophy, and then, along with Frege's theory of sense and reference, became one of the two classical theories of reference.[2] "On Denoting" is now being studied from an historical point of view as arising out of Russell's qualms about his own prior theory of denoting concepts. Like Frege's theory of sense, however, the role of the theory of descriptions in the larger logicist project is not well understood. Frege's theory of sense precedes his foundational work, the *Grundgesetze der Arithmetik*, by only a few years. Yet after the introductory material, senses do not appear in the technical portion of

Grundgesetze, which is occupied with the reference, or *Bedeutung*, in the extensional logic of courses of values (*Werthverlaufe*) of concepts, his logicist version of classes. Frege's theory of sense, it seems, is justifiably foundational in the later development of the philosophy of language, but is not so fundamental to his own life's work, the project of reducing mathematics to logic.

Russell's theory of descriptions might seem to be similarly a digression into the philosophy of language by a philosopher whose main project was to write a long book proving the principles of mathematics from definitions using symbolic logic. My project in this paper is to explain one of the ways that definite descriptions enter into the technical project of *Principia Mathematica*, namely in ∗30 "Descriptive Functions." Descriptive functions are simply ordinary mathematical functions such as the sine function, or addition. Number ∗30 is the origin of the now familiar notion in elementary logic of eliminating functions in favor of relations, and so is part of our conception of elementary logic as ending with the logic of relations, with the addition of complex terms, including function symbols, as an extra, optional development. I wish to argue, however, that this familiar way of reducing logic with functions to the logic of relations alone was in fact a step in Russell's logicist project, one which he took in self conscious opposition to Frege's use of mathematical functions as a primitive notion in his logic. As such "descriptive functions" were important to Russell's reduction of mathematics to logic.

Definite descriptions have an important role in Russell's theory of propositions dating from *Principles of Mathematics* in 1903, where Russell uses the theory of denoting concepts which he only replaced in 1905 with the theory of "On Denoting." Propositions in *Principles*, are composed of "terms" which include individuals and denoting concepts. The predicative constituents of propositions, the terms introduced by predicates, when taken in extension, play the role of classes. These concepts, obviously, are crucial to the logicist account of natural numbers and all other entities that mathematics deals with. The subjects of propositions will be individuals, when it is indeed individuals about which we make judgements, but, more generally, denoting concepts, which enable us to judge about terms with which we are not acquainted, such as infinite classes, and, more familiar from "On Denoting", non-existents, such as the round square or the present King of France. A proper account of definite descriptions, as a special sort of denoting concept, is, then, appropriate in Russell's preliminary, foundational, thinking about the logic to which mathematics is to be reduced. Though appropriate in an account of fundamental notions of logic, however, an account of definite descriptions is not central to the technical development of *Principia Mathematica*, which came to be based on the concept of propositional function, rather than the propositions which are the center of attention in *Principles of Mathematics*.

Russell's theory of definite descriptions is also important for the project of *Principia* as a model of the technique of contextual definition which is used there in ∗20 "Classes," to reduce classes to propositional functions. The theory of descriptions in ∗12 is based on a pair of contextual definitions, which allow the elimination of expressions for definite descriptions from the contexts in which they occur. The primary definition is:

$$*14{\cdot}01.\ \ [(\imath x)\phi x] \,.\, \psi(\imath x)\phi x \,.=:(\exists b):\phi x \,.\equiv_x .\, x=b:\psi b \ \ \text{Df}$$

This can be paraphrased as saying that ⌜the ϕ is ψ⌝ means the same as ⌜There is a b such that anything x is ϕ if and only if that x is identical with b, and that b is ψ⌝. Here ψ is the context from which the description $(\imath x)\phi x$ is to be eliminated. That this is the "scope" of the description is indicated by the prefixed occurrence of the description in square brackets: $[(\imath x)\phi x]$. This definition allows the replacement of formulas in which definite descriptions appear is subject position. A further contextual definition is provided for the occurrence of descriptions as, $E!(\imath x)\phi x$, which expresses the assertion that a description is "proper", that is, that is that there is exactly one ϕ.

The second way in which the theory of definite descriptions enters into the logicist reduction of mathematics in *Principia Mathematica*, is as a model for the similar contextual definition of class expressions. Just as the definitions of ∗14 allow for the elimination of definite descriptions from different contexts, so the theory of classes in ∗20 is based on a series of contextual definitions. Occurrences of class expressions ⌜$\hat{z}(\psi z)$⌝, read as "the class of z which are ψ", can be eliminated from contexts f via the primary definition:

$$*20{\cdot}01\ \ f\{\hat{z}(\psi z)\}.=:(\exists\phi):\phi!x \,.\equiv_x .\, \psi x : f\{\phi!\hat{z}\} \ \ \text{Df}$$

To say that the class $\hat{z}(\psi z)$ is f is to say that there is some (predicative) function ϕ which is coextensive with ψ which is f. There is no explicit mention of scope, but in all regards this definition closely copies that of definite descriptions.[3] The definition of class expressions is completed by a series of other definitions, including those which use variables that range over classes, the "greek letters" such as α, which are used as bound (apparent) and free (real) variables for classes. Together, the definitions of ∗20 provide a reduction of the theory of classes to the theory of propositional functions. One immediate consequence of this definition is that a solution for Russell's paradox is provided by the restrictions of the theory of types. The "class of all classes that are not members of themselves", upon analysis, requires a function to apply to another function of the same type, which is prohibited by the theory of types. (See my Linsky 2002.) While this "no-classes" theory of classes succeeds in resolving the paradoxes via the elimination of talk of classes in favor of talk about propositional functions, it is precisely at this point that we part ways with the now standard, alternative, project of founding mathematics on axiomatic set theory. Rather than rely on the notion of propositional function to explain classes, philosophers who favor axiomatic set theory prefer the first order theory of sets, as formulated in one of the standard axiomatic theories such as that of Zermelo-Fraenkel set theory with the Axiom of Choice, ZFC. Propositional functions, it is felt, are obscure, and not even presented as familiar mathematical functions from arguments to propositions.

The next section of *Principia Mathematica*, ∗21 "General Theory of Relations" presents the extension of the "no-classes" theory to the corresponding notion for binary relations, the theory of "relations in extension." By analogy with the way the no-classes theory of ∗20 defines a class expression $\hat{z}(\psi z)$ using a contextual definition, in ∗21 we are given contextual definitions for eliminating expressions of the form $\hat{x}\hat{y}\ \psi(x,y)$, which represents the "relation in extension" which holds between x and y when $\psi(x,y)$ obtains:

$$*21{\cdot}01.\ \ f\{\hat{x}\hat{y}\ \psi(x,y)\}\,.= \\ :(\exists\phi):\phi\,!(x,y)\,.\equiv_{x,y} .\,\psi(x,y):f\{\phi\,!(\hat{u},\hat{v})\} \ \ \text{Df}$$

The relation of x bearing ψ to y has the property f just in case some predicative function ϕ , which is coextensive with ψ has the property f. From ∗21 on "Capital Latin Letters", i.e. 'R', 'S', 'T', etc., are reserved for these relations in extension. They are variables, replaced by such expressions as $\hat{x}\hat{y}\ \psi\,!\,(x, y)$, "just", Whitehead and Russell say, "... as we used Greek letters for variable expressions of the form $\hat{z}(\phi\,!\,z)$." (*PM* 201) These new symbols for relations in extension are written between variables, as in xRy or uSv. A propositional function would precede the variables, as in $\phi(x, y)$. (It is not clear how this notation for relations in extension would be extended to three or four place relations. Indeed in general below, as when talking about the analysis of relations in terms of sets of ordered pairs, the discussion will always be restricted to binary relations.) It should be noted, as Quine has observed, that the intensional propositional functions represented by ϕ, and ψ, etc., drop out here from the development of *Principia Mathematica*, and that from this point on we only encounter relations in extension, symbolized by 'R', 'S', 'T', etc. (Quine 1963, 251).

Definite descriptions, though of course very important to the later development of the philosophy of language, do not appear explicitly in the later sections of *PM* where in fact the work of reducing mathematics to logic is really carried out. In fact it is after ∗30·01 that description operators, the familiar "rotated iotas", disappear, having, I would argue, performed their most important technical function. We are now ready for the third way in which the theory of definite descriptions enters into the logicist project of *Principia Mathematica*, as key to the definition of "Descriptive Functions", the topic of this paper. This takes the form of yet another definition, in this case of the expression $R‘y$, to be read as "the R of y":

∗30·01. $R‘y = (\iota x)\, xRy$ Df

The expression $R‘y$ is defined by the definite description, $(\iota x)\ xRy$. If xRy means "x is father of y" then $R‘y$ is "the x such that x is father of y", or "the father of y". As Whitehead and Russell point out, this definition is not a contextual definition, which shows how expression $R‘y$ is to be eliminated from a context, such as $\psi(R‘y)$, but simply as an explicit instruction about the replacement of symbols $R‘$, wherever they occur. The notion of "descriptive function" provides an analysis of the ubiquitous "mathematical functions" of arithmetic and analysis which are reduced in later numbers of *Principia Mathematica*. Whitehead and Russell say:

> The functions hitherto considered, with the exception of a few particular functions such as $\alpha \cap \beta$, have been propositional, *i.e.* have had propositions for their values. But the ordinary functions of mathematics, such as x^2, sin x, log x, are not propositional. Functions of this kind always mean "the term having such and such a relation to x." For this reason they may be called *descriptive* functions, because they *describe* a certain term by means of its relation to their argument. Thus "sin $\pi/2$" describes the number 1; yet propositions in which sin $\pi/2$ occurs are not the same as they would be if 1 were substituted for sin $\pi/2$. This appears, *e.g.* from the proposition "sin $\pi/2 = 1$," which conveys valuable information, whereas "$1 = 1$" is trivial. Descriptive functions, like descriptions in general, have no meaning by themselves, but only as constituents of propositions. (*PM*, 231)

Descriptive functions provide *Principia Mathematica*'s analysis of mathematical functions, a Logicist analysis in terms of the logical notions of relation in extension and definite descriptions. It has been said that Frege "mathematicized" logic in preparation for his analysis of arithmetic.[4] That mathematization involved not only the invention of symbolic logic, but also reliance on the mathematical notion of function as a primitive notion in his logic. Concepts are functions from objects to truth values. Frege's notion of the extension of a concept is its course of values, which is a notion that applies to all functions. The notion of course of values is centrally implicated in Russell's paradox, and so is seen, like Whitehead and Russell's theory, as one of the unsuccessful logicist attempts to avoid postulating sets as primitive, mathematical, entities. The account of descriptive functions in ∗30 thus brings out clearly, some might think, the primary objections to Whitehead and Russell's version of logicism. It relies on notions much better understood within the mathematical theory of sets, it is thought. A function, on this account, is simply as set of ordered pairs, ordered pairs themselves being sets of a certain sort, and a propositional function would be a function from arguments to propositions. As propositions are not needed for the extensional, first order, logic in which axiomatic set theory is formulated, ∗30 thus epitomizes the wrong path taken by Whitehead and Russell's version of logicism.

However, I would like to suggest that an examination of the development of the idea of function in logic from Frege and Russell on into the early part of the twentieth century will defend the notion of descriptive function as a successful way of reducing the mathematical notion of function to logical notions alone. With the exception of a few remarks, from Russell's notes on Frege's works, and his unpublished paper "On Meaning and Denotation" from 1903, this review relies on material in *Principles of Mathematics* and its Appendix A, "On the Doctrines of Frege", but presented with a different emphasis than is usual. In particular, the emphasis will be on customary mathematical functions such as the sine function, or addition, and their reduction in axiomatic set theory to sets of ordered pairs, and less with the more idiosyncratic sorts of functions, such as concepts and propositional functions.

Frege on Functions

While it is correct to say that Frege relies on the notion of mathematical function as a primitive, that is not to say that he did not provide a famously original and ground breaking logical analysis of function expressions and variables. Frege's 1891 paper "Function and Concept" and most explicitly his 1904 paper "What is a Function?" talk about the mathematical notion of function, of which concepts are a special case. Frege explains the nature of variables as being linguistic entities which may be assigned different values and not as signs of "variable quantities" as many had confusedly described them to be. Frege's further notion of concepts as "unsaturated entities" which are completed by objects and yield truth values is well known. A function in general, and mathematical functions among them, will also refer to unsaturated entities which yield objects as values. A function expression, then, such as $\sin x$, x^2, and $\log x$ will have as its *Bedeutung*, or reference an unsaturated entity which, when applied to a number as argument, yields a number as value. The logical status of expressions for functions is that they are "incomplete" names for numbers. Just as Frege had problems in even naming concepts such as "the concept horse", similarly there is a difficulty with naming functions.[5] In fact the sine function ought to be expressed somehow as 'sin( )' with a blank or hole to indicate its unsaturated nature. The expression '$\sin x$' really expresses a given number, the value of the function, for each assignment of a number to the variable x. It is clear from the discussion of the problem of naming concepts that Frege would have rejected Church's lambda notation as a way of naming functions, for example, with '$\lambda x \sin x$' as naming the sine function.

Russell's views on this "concept horse" problem are in the Appendix to *Principles of Mathematics*. In §481 Russell agrees with Frege that it is just "...some terms can only occur as subjects...", in opposition to Kerry's view that "Begriffe also can occur as subjects ...", but goes on to disagree with the further claim that they are subjects standing "in the same relation" to their predicates.

> But he [Frege] goes on to make a second point that appears mistaken. We can, he says, have a concept falling under a higher one (as Socrates falls under man, he means, not as Greek falls under man): but in such cases, it is not the concept itself, but its name, that is in question (BuG. p. 195). "The concept horse," he says, is not a concept, but a thing; the peculiar use is indicated by inverted commas (*ib.* p.196). But a few pages later he makes statements which seem to involve a different view. A concept, he says, is essentially predicative even when something is asserted of it: an assertion which can be made of a concept does not fit an object. When a thing is said to fall under a concept, and when a concept is said to fall under a higher concept, the two relations involved, though similar, are not the same (*ib.* p.201). It is difficult to me to reconcile these remarks with those of p.195; but I shall return to this point shortly. (PoM, 507)

On the next page Russell discusses what is essentially the difference in logical type between objects and concepts:

> Another point of difference from Frege, in which, however, he appears in the right, lies in the fact that I place no restriction upon the variation of the variable, whereas Frege, according to the nature of the function, confines the variable to things, functions of the first order of one variable, functions of the first order with two variables, functions of

> the second order with one variable, and so on. There are thus for him an infinite number of different kinds of variability. This arises from the fact that he regards as distinct the concept occuring as such and the concept occurring as term, which I (§49) have identified. For me, the functions, which cannot be values of variables in functions of the first order, are non-entities and false abstractions. (PoM, 508-9)

Russell's remark that Frege is "in the right" on this issue has to do with the division of propositional functions into types. Russell says that "The contradiction discussed in Chapter X seems to show that some mystery lurks in the variation of propositional functions; but for the present Frege's theory of different kinds of variables must, I think, be accepted." (PoM, 510).

Russell returns to the "concept horse problem" in §483, arguing that Frege is simply wrong, and that concepts can be subjects of propositions. He says:

> Frege, it may be observed, does not seem to have clearly disentangled the logical and linguistic elements of naming: the former depend upon denoting, and have, I think, a much more restricted range than Frege allows them. (PoM, 510)

This is a long way from the contemporary view that functions are simply sets of ordered pairs. In his *Introduction to Mathematical Logic*, Alonzo Church manages to turn Frege's view into the current standard current view on the logical syntax of function expressions and terms. Church avoids Frege's talk of function expressions having as a reference (Bedeutung) some unsaturated (and un-nameable) entity, which, when saturated by an argument, gives a value. Instead we find:

> If we suppose the language fixed, every singular form (function expression) has corresponding to it a function f (which we will call the *associated function* of the form) by the rule that the value of f for an argument x is the same as the value of the form for the value x of the free variable of the form . . . (Church 1956, 19)

This account avoids the expressions "denoted" or "designates", instead using the neutral "corresponding to", and "associated with". Church

wishes to explain the semantics of function expressions without running afoul of Frege's "concept horse" problem by saying that functional expressions *name* functions. But this is Frege's account of the semantics. Church, and those after him for some time, took the difference in kind between functions and objects, as a difference of logical type. It was only in the late 1930s that, following Quine, it became standard to view logic as first order logic, and relations and functions, via their reduction to sets of ordered pairs, as themselves just objects.[6]

If we look carefully at Basic Law V of the *Grundgesetze*, the law that leads to the paradox, we see that it is actually about functions in general:

$$\vdash (\grave{\epsilon} f(\epsilon) = \grave{\alpha} g(\alpha)) = (\forall \mathfrak{a} f(\mathfrak{a}) = g(\mathfrak{a}))$$

The expression $\grave{\epsilon} f(\epsilon)$ has as *Bedeutung* or referent, the *Werthverlauf*, or "course of values" of the function f. Basic Law V thus says that the course of values of f is the same as the course of values of the function g just in case the values of f and g are the same for every argument $\mathfrak{a}$. In the case that f is a function from objects to truth values, and so a concept, the course of values is naturally seen as the extension of the concept, as a class. But for any other sort of function, the course of values is an object, like the *graph* of the function, the set of pairs of arguments and values. For the special case of concepts, Basic Law V does say that extensions are the same when concepts are coextensive. It is natural, then, that Russell saw (mathematical) functions as figuring prominently in Frege's account of terms and so in his logic.

Function expressions for Frege will have a sense as well, although he does not discuss this in much detail. That sense will provide a connection between the argument of a function and its value, presumably, in the way that the sense of a name provides a "route to the reference" of the name. While mathematical functions simply map numbers onto numbers, there is still some notion of the connection between the two, as embodied in the sense of the function expression. Russell, in the margin of *Grundgesetze §2* writes "What is the *Sinn* of $\xi^2 = 4$? This is a most puzzling question."[7]

It is possible that the notion of the sense of an equation may be at the heart of Frege's use of the Sinn/Bedeutung distinction in logic. For if a functional term expression like 'ξ^2' simply has as its Bedeutung the value of a function for an argument, thus for the argument 2 the expression ‘2^2’ is simply another name for 2, and, furthermore there is no trace of the argument (or function) in the value, 4, then the equation ‘$2^2 = 4$’ is a trivial identity. ‘$2^2 = 4$’ is then not different from ‘$4 = 4$.’ If expressions such as ‘$2^2 = 4$’ are to be derived from logical principles alone, and this is to reveal something about the status of arithmetical truths, then there must be more to the derivations than a string of self identities (or names for the True, which is, after all, the Bedeutung of each logical truth). So, one might see the attention to identity sentences at the beginning of “Ueber Sinn und Bedeutung” as not just an sample of a problem picked almost at random, the attention being on the replacement of names with the same reference in sentences ‘a = ...’, but really as directed at *identities*, as important to the theory of mathematical functions, and so for a defense of the epistemological interest in mathematics, if it is indeed devoted to sentences and other expressions with the Bedeutung that Frege says they have. Oddly, then, Russell’s seemingly naive question, “What is the *Sinn* of $\xi^2 = 4$?”, gets at the very point of the theory of sense, to justify the account of function expressions, that Frege relied on.

In any case, then, however it is that Frege provides a logical analysis of functions, including both his notion of unsaturated entities, and the notion of sense, this account is in aid of understanding the functions with which mathematicians were already familiar.

Russell’s criticisms of Frege

There is little direct textual support for my thesis that Russell’s dissatisfaction with Frege’s notion of function was due to its being insufficiently logicist. In Russell’s early writings there is little attention to a demarcation between logic and mathematics, or attention to whether a notion is logical or not.[8] Instead, Russell’s attention is always on finding the proper logical analysis of a notion, so that any successful analysis is automatically a logicist account. There are objections to Frege’s theory of functions, however, expressed in Appendix A to *Principles of Mathematics*, and they can be read in this light. Thus we have:

> The fundamental case is that where our unity is a propositional concept. From this is derived the usual mathematical notion of a function, which might at first sight seem simpler. If $f(x)$ is not a propositional function, its value for a given value of x ($f(x)$ being assumed to be one-valued) is the term y satisfying the propositional function $y = f(x)$, *i.e.* satisfying, for the given value of x, some relational proposition; this relational proposition is involved in the definition of $f(x)$, and some such propositional function is required in the definition of any function which is not propositional. (*PoM*, 508)

Russell here asserts that the notion of denoting, and hence of descriptive function, is presupposed in the mathematical notion of a function in the expression of *the* value of a function. Specific criticisms of the account of functions follow on the next page:

> Frege’s general definition of a function, which is also intended to cover also functions which are not propositional, may be shown to be inadequate by considering what may be called the identical function, *i.e.* x as a function of x. If we follow Frege’s advice, and remove x in hopes of having the function left, we find that nothing is left at all; yet nothing is not the meaning of the identical function. (PoM, 509)

The objection is that Frege’s metaphor for incompleteness, the gap in a denoting expression cannot account for the identity function, which takes x as an argument and returns x as a value. An equation, ‘$f(x) = x$’ can express such a function, but an expression directly denoting the value, with the argument deleted. But, Russell argues, equations presuppose a denoting concept, “the” value of a function. Russell continues his criticisms further on that page:

> Frege wishes to have the empty places where the argument is to be inserted indicated in some way; thus he says that in $2x^3 + x$ the function is $2(\ )^3 + (\ )$ But here his requirement that the two empty places be filled by the same letter cannot be indicated; there is no way of distinguishing what we mean from the function involved in $2x^3 + y$. (PoM, 509)

Frege's talk of expressions for functions as incomplete, suggested by a hole or empty spot, doesn't explain the role of variables in function expressions. Later, with Church's lambda calculus, it was clear that the variables in functional expressions are to be seen as bound variables. $\lambda x\ 2x^3 + x$ is clearly distinguished from $\lambda x \lambda y\ 2x^3 + y$. There is more to the logical analysis of function expressions than the unsaturatedness of functions.[9]

> The fact seems to be that we want the notion of any term of a certain class, and that this is what our empty places really stand for. The relation, as a single entity, is the relation (6) . . . above [the relation of the member of the class . . . to the value which the variable has in that member]; we can then consider any relatum of this relation, or the assertion of all or some of the relata, and any relation can be expressed in terms of the corresponding referent, as "Socrates is a man" is expressed in terms of Socrates. But the usual formal apparatus of the calculus of relations cannot be employed, because it presupposes propositional functions. We may say that a propositional function is a many-one relation which has all terms for the class of its referents, and has its relata contained among propositions: or, if we prefer, we may call the class of relata of such a relation a propositional function. But the air of formal definition about these statements is fallacious, since propositional functions are presupposed in defining the class of referents and relata of a relation. (PoM, 509)

Russell here objects to saying that propositional functions are functions from individuals to propositions, on the grounds that that is a circular account, "since propositional functions are presupposed in defining the class of referents and relata of a relation." But he also suggests that the explanation of variables in functions expressions involves denoting as well. The "empty places" in a function expression really stand for "any term" of a certain class. "Any" is one of the class of denoting expressions analyzed in "On Denoting", with variables themselves remaining as among the last, unanalyzed, denoting expressions when definite and indefinite descriptions have been eliminated. Russell here argues that the role of variables in function expressions is understood if they are analyzed using propositional functions.

Since Frege holds that function expressions are simply incomplete denoting expressions, we can also look to Russell's objections to Frege's theory of descriptions and other "denoting expressions" in "On Denoting" and elsewhere to see his other objections. Aside from the problems of the "Gray's Elegy" argument, which seem to have something to do with the problem of referring to functions, the main problem with Frege's view is the difficulties with improper descriptions. In fact, as I have argued with F.J. Pelletier (Pelletier & Linsky 2005), it is not clear which of four different theories of improper descriptions is Frege's official view. Thus I would conclude that despite the accomplishments of Frege's papers in explaining the logical status of function expressions, it was Russell's dissatisfaction with that very analysis, centering on the problem of improper descriptions, that embodies his objections to the priority, or primitive status, of mathematical functions.

Rather than simply giving an account of partial functions, so that 'dividing by 0' is treated like referring to the present king of France, in fact the notion of descriptive function plays a more important role in Russell's logicism, in that it allows the reduction of a mathematical notion, still primitive in Frege's work, to logical notions. While perhaps puzzling to our eyes, when compared with mathematical functions, Russell's propositional functions were central to his logic, and so, I will argue, central to his logicism.

Russell's views about the relation between mathematical functions and propositional functions, or relations, are not primarily driven by a reaction to Frege. They seem to have been independently motivated and to have been developed before Russell's more careful encounter with Frege in the summer of 1902. Consider the following from "On Meaning and Denotation", from 1903:

> If we take denoting to be fundamental, the natural way to assert a many-one relation will not be xRy but $y = \phi x$. This, of course, is the usual mathematical way; and there is much to be said for it. All the ordinary functions, such as x^2, sin x, log x, etc., seem to occur more naturally in this form than as $\imath\breve{R}|x$. Again, in ordinary language, "y is the father of x" clearly states an identity, not a relation: it is "y

= the father of x". (CP4, 340)

> But if we take *propositional* functions to be fundamental – as I have always done, first consciously and then unconsciously – we must proceed through relations to get to ordinary functions. For then we start with ordinary functions such as "x is a man"; these are originally the only functions of one variable. To get at functions of another sort, we have to pass through xRy; but then, with $\imath$, we get all the problems of denoting. And, as we have seen, a a form of denoting more difficult than $\imath$ is involved in the use of variables to start with. Thus denoting seems impossible to escape from. (CP4, 340)

So, Russell does see propositional functions, or rather, relations, as more fundamental than mathematical functions. Indeed, he adopted this position so surely that it became "unconscious" at some point. However, Russell sees the move to relations as problematic, requiring a proper account of denoting. So, although Russell may have found propositional functions to be more basic than mathematical functions, until he solved the problem of denoting, (in "On Denoting" in 1905), he was not justified in thinking that he had explained the less obvious in terms of the more basic, instead the reduction of mathematical functions led directly to his big problem that concerned him in those days, the problem of denoting.

With a proper theory of denoting, in particular, the theory of descriptions of ∗12 of *Principia Mathematica*, in hand, Whitehead and Russell are then ready to complete the logicist analysis of mathematical functions as "descriptive functions" in ∗30.

Notes

[1] Thanks to Allen Hazen, James Levine, Paul Oppenheimer and Ed Zalta for discussions of the paper, and to the participants in the Riga conference, in particular my co-symposiasts James Levine and Mike Beaney. A companion essay, "From Descriptive Functions to Sets of Ordered Pairs", was presented at the 31st International Wittgenstein Symposium in August 2008, and in the volume *Reduction and Elimination in Philosophy and the Sciences*, Alexander Hieke and Hannes Leitgeb eds., Ontos Verlag, 2009.

[2] Ramsey (1931, 263 n) first called it a "paradigm of Philosophy".

[3] Leon Chwistek paid attention to the role of scope in the no-classes theory and discussed it in his paper "The Theory of Constructive Types", see my (Linsky 2009)

[4] As by Burton Dreben, according to Peter Hylton (1993, n. 28).

[5] Frege introduces this problem in "Function and Concept", (Frege 1891, 196).

[6] See (Mancosu 2005, 335-9).

[7] See (Linsky 2004, 14).

[8] Thus his first reaction to seeing Frege's analysis of the ancestral in purely logical terms was simply to call it "…ingenious: it is better than Peano's induction." (Linsky 2004-5, 137). Thus what we see as a logicist account of induction to the inheritance of hereditary properties was described by Russell as merely *better* than assuming an axiom of induction. Still, Russell adopted Frege's analysis immediately, and later described it as an essential step in the logicist program.

[9] Philip Ebert has pointed out that in *Grundgesetze*, Frege uses the Greek letters ξ and ζ for just this purpose. Indeed Russell copies this notation in his question "What is the *Sinn* of $\xi^2 = 4$?". Clearly Russell is criticizing the use of parentheses around a blank space in "Function and Concept", and making the point that the notion of unsaturatedness alone will not account for all the properties of functions of several arguments.

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